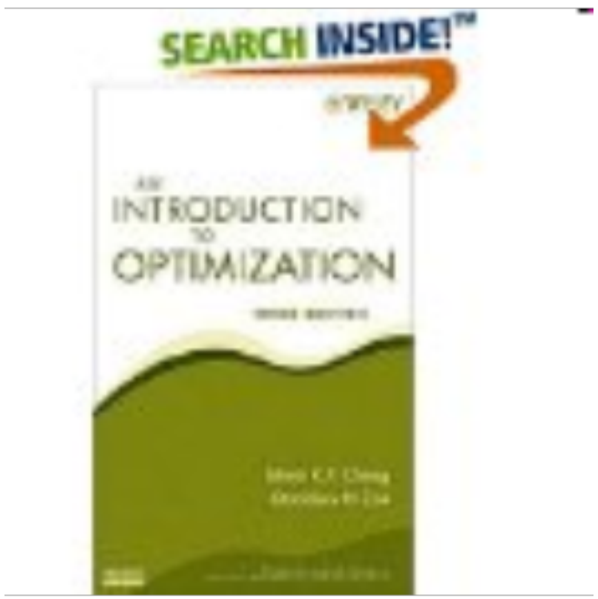
Part II
Linear Constrained Optimization
Chapter 17
DUALITY



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Dual Linear Programs

Table 17.1 Symmetric Form of Duality

Primal (1)		Dual (2)	
minimize	$c^T x$	maximize	$\lambda^T b$
subject to	$Ax \geq b$	subject to	$\lambda^T A \leq c^T$
	$x \geq 0$		$\lambda \geq 0$

$A^T \lambda \leq c$

$\min \ (-b^T)\lambda$
s.t. $(-A^T)\lambda \geq -c$
$\lambda \geq 0$

$\max \ x^T(-c)$
s.t. $x^T(-A^T) \leq -b^T$
$x \geq 0$

$\min \ c^T x$
s.t. $Ax \geq b$
$x \geq 0$

Table 17.2 Asymmetric Form of Duality

Primal (3)		Dual (4)	
minimize	$c^T x$	maximize	$\lambda^T b$
subject to	$Ax = b$	subject to	$\lambda^T A \leq c^T$
	$x \geq 0$		

Dual Problem for an LP in Standard Form

- Standard form

$$\begin{aligned} &\text{minimize } c^T x \\ &\text{subject to } Ax = b \\ &\qquad x \geq 0. \end{aligned}$$

$$\begin{aligned} \boldsymbol{Ax} &\geq \boldsymbol{b} \\ -\boldsymbol{Ax} &\geq -\boldsymbol{b}. \end{aligned}$$

$Ax \le b$

$$\begin{aligned} &\text{minimize} && \boldsymbol{c}^T \boldsymbol{x} \\ &\text{subject to} && \begin{bmatrix} \boldsymbol{A} \\ -\boldsymbol{A} \end{bmatrix} \boldsymbol{x} \geq \begin{bmatrix} \boldsymbol{b} \\ -\boldsymbol{b} \end{bmatrix} \\ & && \boldsymbol{x} \geq \boldsymbol{0}. \end{aligned}$$

$m \quad m$

$$\begin{aligned} &\text{maximize} && [\boldsymbol{u}^T \ \boldsymbol{v}^T] \begin{bmatrix} \boldsymbol{b} \\ -\boldsymbol{b} \end{bmatrix} \\ &\text{subject to} && [\boldsymbol{u}^T \ \boldsymbol{v}^T] \begin{bmatrix} \boldsymbol{A} \\ -\boldsymbol{A} \end{bmatrix} \leq \boldsymbol{c}^T \\ & && \boldsymbol{u}, \boldsymbol{v} \geq \boldsymbol{0}. \end{aligned}$$

$u^T b - v^T b = (u^T - v^T) b$

$u^T A - v^T A$

$$\begin{aligned} &\text{maximize} && (\boldsymbol{u} - \boldsymbol{v})^T \boldsymbol{b} \\ &\text{subject to} && (\boldsymbol{u} - \boldsymbol{v})^T \boldsymbol{A} \leq \boldsymbol{c}^T \\ & && \boldsymbol{u}, \boldsymbol{v} \geq \boldsymbol{0}. \end{aligned}$$

$\boldsymbol{\lambda} = \boldsymbol{u} - \boldsymbol{v}$ and $\boldsymbol{u}, \boldsymbol{v} \geq \boldsymbol{0}$,

$$\begin{aligned} &\text{maximize } \lambda^T b \\ &\text{subject to } \lambda^T A \leq c^T. \end{aligned}$$

Example

- Write the dual of the following linear programming problem:

$$\text{minimize } c^T x$$

$$\text{subject to } Ax \geq b.$$

max $-c^T x$ $n \times 1$

s.t $-Ax \leq -b$ $m \times 1$

⇓

min $-b^T y$ $m \times 1$

s.t $-A^T y \geq -c$

$y \geq 0$

max $b^T y$

s.t $A^T y \leq c$

$y \geq 0$

$y^T A \leq c^T$

$1 \times m$ $m \times n$ $1 \times n$

Example: Dual of the Diet Problem

x_j the amount of the jth food consumed.

c_j the cost per unit of the jth food required.

$$\begin{array}{lrcl} \text{minimize} & c_1x_1 + c_2x_2 + \cdots + c_nx_n & & \\ \text{subject to} & a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n & \geq & b_1 \\ & a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n & \geq & b_2 \\ & & \vdots & \\ & a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n & \geq & b_m \\ & x_1, \ldots, x_n & \geq & 0. \end{array}$$

b_i the amount of the ith nutrient required.

a_{ij} the amount of the ith nutrient per unit of the jth food..

λ_i the price of the ith nutrient pill.

$$\begin{array}{lrcl} \text{maximize} & \lambda^T b & & \\ & \lambda_1 a_{11} + \cdots + \lambda_m a_{m1} & \leq & c_1 \\ & & \vdots & \\ & \lambda_1 a_{1n} + \cdots + \lambda_m a_{mn} & \leq & c_n. \\ & & \lambda \geq 0. & \end{array}$$

n food

m nutrients

A ($m \times n$) composing matrix

Example 17.2 Consider the following linear programming problem:

$$\begin{array}{lrcl} \text{maximize} & 2x_1 + 5x_2 + x_3 & & \\ \text{subject to} & 2x_1 - x_2 + 7x_3 & \leq & 6 \\ & x_1 + 3x_2 + 4x_3 & \leq & 9 \\ & 3x_1 + 6x_2 + x_3 & \leq & 3 \\ & x_1, x_2, x_3 & \geq & 0. \end{array}$$

Find the corresponding dual problem and solve it.

$= \lambda^T b$ $\quad b = \begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix}$, $c = \begin{bmatrix} 6 \\ 9 \\ 3 \end{bmatrix}$, $A = \begin{bmatrix} 2 & -1 & 7 \\ 1 & 3 & 4 \\ 3 & 6 & 1 \end{bmatrix}$

min $6y_1 + 9y_2 + 3y_3 = c^T y$

s.t $\begin{bmatrix} 2 & 1 & 3 \\ -1 & 3 & 6 \\ 7 & 4 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \geq \begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix}$ $\quad A^T x \leq c$

$y_1, y_2, y_3 \geq 0$

$r_D = c_D^T - c_B^T B^{-1} D$

phase 2

Properties of Dual Problems

Lemma 17.1 ***Weak Duality Lemma.*** *Suppose that* $\mathbf{x}$ *and* $\boldsymbol{\lambda}$ *are feasible solutions to primal and dual LP problems, respectively (either in the symmetric or asymmetric form). Then,* $\mathbf{c}^T\mathbf{x} \geq \boldsymbol{\lambda}^T\mathbf{b}$. □

Proof : (Hint) $c^T x - \lambda^T b$

$\geq \lambda^T A x - \lambda^T b$

$\geq \lambda^T (Ax - b) \geq 0$

$Ax \geq b$

$x \geq 0$

$\lambda^T A x \leq c^T x$

$\lambda \geq 0$

$Ax = b$

symmetric

asymmetric

→If the cost of one of the problems is unbounded, the other problem has no feasible solution.

Theorem 17.1 *Suppose that* $\boldsymbol{x}_0$ *and* $\boldsymbol{\lambda}_0$ *are feasible solutions to the primal and dual, respectively (either in symmetric or asymmetric form). If* $\boldsymbol{c}^T\boldsymbol{x}_0 = \boldsymbol{\lambda}_0^T\boldsymbol{b}$*, then* $\boldsymbol{x}_0$ *and* $\boldsymbol{\lambda}_0$ *are optimal solutions to their respective problems.* □

Proof : (Hint) $c^T x \geq \lambda_0^T b = c^T x_0$

$\forall x$ prime $\geq$ dual

Duality Theorem

$$\begin{bmatrix} I_m & B^{-1}D & B^{-1}b \\ 0^T & c_D^T - c_B^T B^{-1} D & -c_B^T B^{-1} b \end{bmatrix}$$

$r_D^T =$

λ^T

Theorem 17.2 ***Duality Theorem.*** *If the primal problem (either in symmetric or asymmetric form) has an optimal solution, then so does the dual, and the optimal values of their respective objective functions are equal.* □

Proof: (For asymmetric)

(1) Find solution to dual: $\lambda^T = c_B^T B^{-1}$

(2) Prove $\lambda^T b = c^T x$

(For symmetric)

Convert the primal into the standard form by adding surplus variables

Obtaining the optimal solution to the dual

$$\begin{bmatrix} \boldsymbol{A} & \boldsymbol{b} \\ \boldsymbol{c}^T & 0 \end{bmatrix} = \begin{bmatrix} \boldsymbol{B} & \boldsymbol{D} & \boldsymbol{b} \\ \boldsymbol{c}_B^T & \boldsymbol{c}_D^T & 0 \end{bmatrix}.$$

$$\begin{bmatrix} \boldsymbol{I} & \boldsymbol{B}^{-1}\boldsymbol{D} & \boldsymbol{B}^{-1}\boldsymbol{b} \\ \boldsymbol{0}^T & \boldsymbol{r}_D^T & -\boldsymbol{c}_B^T\boldsymbol{B}^{-1}\boldsymbol{b} \end{bmatrix},$$

$r_D^T = c_D^T - c_B^T B^{-1} D$

Therefore, if we define $\boldsymbol{r}^T = [\boldsymbol{0}^T, \boldsymbol{r}_D^T]$, then combining the equations $\boldsymbol{\lambda}^T\boldsymbol{D} = \boldsymbol{c}_D^T - \boldsymbol{r}_D^T$ and $\boldsymbol{\lambda}^T\boldsymbol{B} = \boldsymbol{c}_B^T$ yields

$$\boldsymbol{\lambda}^T\boldsymbol{A} = \boldsymbol{c}^T - \boldsymbol{r}^T.$$

If rank D=m, then we can solve for λ using the vector r_D

Example 17.4 In Example 17.2, the tableau for the primal in standard form is

	a_1	a_2	a_3	a_4	a_5	a_6	b
	2	−1	7	1	0	0	6
	1	3	4	0	1	0	9
	3	6	1	0	0	1	3
c^T	−2	−5	−1	0	0	0	0

	$\frac{15}{43}$	0	1	$\frac{6}{43}$	0	$\frac{1}{43}$	$\frac{39}{43}$
	$-\frac{74}{43}$	0	0	$-\frac{21}{43}$	1	$-\frac{25}{43}$	$\frac{186}{43}$
	$\frac{19}{43}$	1	0	$-\frac{1}{43}$	0	$\frac{7}{43}$	$\frac{15}{43}$
r^T	$\frac{24}{43}$	0	0	$\frac{1}{43}$	0	$\frac{36}{43}$	$\frac{114}{43}$

We can now find the solution of the dual from the above simplex tableau using the equation $\boldsymbol{\lambda}^T \boldsymbol{D} = \boldsymbol{c}_D^T - \boldsymbol{r}_D^T$:

$$[\lambda_1, \lambda_2, \lambda_3] \begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 0 \\ 3 & 0 & 1 \end{bmatrix} = [-2, 0, 0] - \left[\frac{24}{43}, \frac{1}{43}, \frac{36}{43}\right].$$

$$\boldsymbol{\lambda}^T = \left[-\frac{1}{43}, 0, -\frac{36}{43}\right],$$

Summary of Prime and Dual

- If one has unbounded objective function values, the other has no feasible solution
- If one has an optimal feasible solution, then so does the other.
- If one has no feasible solution, then
  - Either has no feasible solution
  - Or has unbounded objective function

Theorem 17.3 ***Complementary Slackness Condition.*** *The feasible solutions $\boldsymbol{x}$ and $\boldsymbol{\lambda}$ to a dual pair of problems (either in symmetric or asymmetric form) are optimal if and only if*

1. $(\boldsymbol{c}^T - \boldsymbol{\lambda}^T \boldsymbol{A})\boldsymbol{x} = 0$; *and*
2. $\boldsymbol{\lambda}^T(\boldsymbol{A}\boldsymbol{x} - \boldsymbol{b}) = 0$.

Proof. We first prove the result for the asymmetric case. Note that condition 2 holds trivially for this case. Therefore, we only consider condition 1.

$\Rightarrow$: If the two solutions are optimal, then by Theorem 17.2, $\boldsymbol{c}^T\boldsymbol{x} = \boldsymbol{\lambda}^T\boldsymbol{b}$. Because $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$, we also have $(\boldsymbol{c}^T - \boldsymbol{\lambda}^T\boldsymbol{A})\boldsymbol{x} = 0$.

$\Leftarrow$: If $(\boldsymbol{c}^T - \boldsymbol{\lambda}^T\boldsymbol{A})\boldsymbol{x} = 0$, then $\boldsymbol{c}^T\boldsymbol{x} = \boldsymbol{\lambda}^T\boldsymbol{A}\boldsymbol{x} = \boldsymbol{\lambda}^T\boldsymbol{b}$. Therefore, by Theorem 17.1, $\boldsymbol{x}$ and $\boldsymbol{\lambda}$ are optimal.

We now prove the result for the symmetric case.

$\Rightarrow$: We first show condition 1. If the two solutions are optimal, then by Theorem 17.2, $\boldsymbol{c}^T\boldsymbol{x} = \boldsymbol{\lambda}^T\boldsymbol{b}$. Because $\boldsymbol{A}\boldsymbol{x} \geq \boldsymbol{b}$ and $\boldsymbol{\lambda} \geq \boldsymbol{0}$, we have

$$(\boldsymbol{c}^T - \boldsymbol{\lambda}^T\boldsymbol{A})\boldsymbol{x} = \boldsymbol{c}^T\boldsymbol{x} - \boldsymbol{\lambda}^T\boldsymbol{A}\boldsymbol{x} = \boldsymbol{\lambda}^T\boldsymbol{b} - \boldsymbol{\lambda}^T\boldsymbol{A}\boldsymbol{x} = \boldsymbol{\lambda}^T(\boldsymbol{b} - \boldsymbol{A}\boldsymbol{x}) \leq 0.$$

On the other hand, since $\boldsymbol{\lambda}^T\boldsymbol{A} \leq \boldsymbol{c}^T$ and $\boldsymbol{x} \geq \boldsymbol{0}$, we have $(\boldsymbol{c}^T - \boldsymbol{\lambda}^T\boldsymbol{A})\boldsymbol{x} \geq 0$. Hence, $(\boldsymbol{c}^T - \boldsymbol{\lambda}^T\boldsymbol{A})\boldsymbol{x} = 0$. To show condition 2, note that since $\boldsymbol{A}\boldsymbol{x} \geq \boldsymbol{b}$ and $\boldsymbol{\lambda} \geq \boldsymbol{0}$, we have $\boldsymbol{\lambda}^T(\boldsymbol{A}\boldsymbol{x} - \boldsymbol{b}) \geq 0$. On the other hand, since $\boldsymbol{\lambda}^T\boldsymbol{A} \leq \boldsymbol{c}^T$ and $\boldsymbol{x} \geq \boldsymbol{0}$, we have $\boldsymbol{\lambda}^T(\boldsymbol{A}\boldsymbol{x} - \boldsymbol{b}) = (\boldsymbol{\lambda}^T\boldsymbol{A} - \boldsymbol{c}^T)\boldsymbol{x} \leq 0$.

$\Leftarrow$: Combining conditions 1 and 2, we get $\boldsymbol{c}^T\boldsymbol{x} = \boldsymbol{\lambda}^T\boldsymbol{A}\boldsymbol{x} = \boldsymbol{\lambda}^T\boldsymbol{b}$. Hence, by Theorem 17.1, $\boldsymbol{x}$ and $\boldsymbol{\lambda}$ are optimal. ∎

Example 17.5 Suppose you have 26 dollars and you wish to purchase some gold. You have a choice of four vendors, with prices (in dollars per ounce) of $1/2, 1, 1/7$, and $1/4$, respectively. You wish to spend your entire 26 dollars by purchasing gold from these four vendors, where x_i is the dollars you spend on vendor i, $i = 1, 2, 3, 4$.

a. Formulate the linear programming problem (in standard form) that reflects your desire to obtain the maximum weight in gold.

b. Write down the dual of the linear programming problem in part a, and find the solution to the dual.

c. Use the complementary slackness condition together with part b to find the optimal values of $x_1, \ldots, x_4$.

DUAL LINEAR PROGRAMS

Table 17.1 Symmetric Form of Duality

Primal	Dual
minimize $c^T x$ subject to $Ax \geq b$ $x \geq 0.$	maximize $b^T y$ subject to $A^T y \leq c$ $y \geq 0.$

Table 17.2 Asymmetric Form of Duality

Primal	Dual
minimize $c^T x$ subject to $Ax = b$ $x \geq 0.$	maximize $b^T y$ subject to $A^T y \leq c.$