7.4 Diagonalization

Fact 7.4.1 The matrix of a linear transformation with respect to an eigenbasis is diagonal

Consider a transformation $T\vec{x} = A\vec{x}$, where A is an $n \times n$ matrix. Suppose B is an eigenbasis for T consisting of vectors $\vec{v}_1, \vec{v}_2, ..., \vec{v}_n$, with $A\vec{v}_i = \lambda_i \vec{v}_i$. Then the B-matrix D of T is

$$D = S^{-1}AS = \begin{bmatrix} \lambda_1 & 0 & . & 0 \\ 0 & \lambda_2 & . & 0 \\ . & . & . & . \\ 0 & 0 & 0 & \lambda_n \end{bmatrix}$$

Here

$$S = \begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & ... & \vec{v}_n \\ | & | & & | \end{bmatrix}$$

$$\begin{array}{ccc} \vec{x} & \overset{A}{\longrightarrow} & A\vec{x} \\ S\uparrow & & \uparrow S \\ \left[\ \vec{x}\ \right]_S & \underset{D}{\longrightarrow} & \left[\ A\vec{x}\ \right]_S \end{array} \qquad \begin{array}{l} \vec{x} = S\left[\ \vec{x}\ \right]_S \\ \left[\ \vec{x}\ \right]_S = S^{-1}\vec{x} \end{array}$$

Def 7.4.2 Diagonalizable matrices
An $n \times n$ matrix A is called *diagonalizable* if A is similar to a diagonal matrix D, that is, if there is an invertible $n \times n$ matrix S such that $D = S^{-1}AS$ is diagonal.

Fact 7.4.3
Matrix A is diagonalizable iff there is an eigenbasis for A. In particular, if an $n \times n$ matrix A has n distinct eigenvalues, then A is diagonalizable.

Alg 7.4.4 Diagonalization

Suppose we are asked to decide whether a given $n \times n$ matrix A is diagonalizable, if so, to find an invertible matrix S such that $S^{-1}AS$ is diagonal. We proceed as follows:

1. Find the eigenvalues of A, i.e., solve $f(\lambda) = det(\lambda I_n - A) = 0$.

2. For each eigenvalue λ, find a basis of the eigenspace $E_\lambda = ker(\lambda I_n - A)$.

3. A is diagonalizable iff the dimensions of the eigenspaces add up to n. In this case, we find an eigenbasis $\vec{v}_1, \vec{v}_2, ..., \vec{v}_n$ for A by combining the bases of the eigenspaces. Let $S = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & ... & \vec{v}_n \end{bmatrix}$, then the matrix $S^{-1}AS$ is a diagonal matrix.

Example Diagonalize the matrix

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Solution

a. The eigenvalues are 0 and 1.

b. $E_0 = ker(A) = span(\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix})$

and $E_1 = ker(I_3 - A) = span \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

c. If we let

$$S = \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

then

$$D = S^{-1}AS = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Alg 7.4.5 Powers of a diagonalizable matrix

To compute the powers A^t of a diagonalizable matrix A (where t is a positive integer), proceed as follows:

1. Use Alg 7.4.4 to diagonalize A, i.e. find S such that $S^{-1}AS = D$.

2. Since $A = SDS^{-1}$, $A^t = SD^tS^{-1}$.

3. To compute D^t, raise the diagonal entries of D to the tth power.