7.2 FINDING THE EIGENVALUES OF A MATRIX

Consider an $n \times n$ matrix A and a scalar λ. By definition λ is an eigenvalue of A if there is a nonzero vector $\vec{v}$ in R^n such that

$$A\vec{v} = \lambda\vec{v}$$

$$\lambda\vec{v} - A\vec{v} = \vec{0}$$

$$(\lambda I_n - A)\vec{v} = \vec{0}$$

An an eigenvector, $\vec{v}$ needs to be a nonzero vector. By definition of the kernel, that

$$ker(\lambda I_n - A) \neq \{\vec{0}\}.$$

(That is, there are other vectors in the kernel besides the zero vector.)
Therefore, the matrix $\lambda I_n - A$ is **not invertible**, and det$(\lambda I_n - A)$=0.

Fact 7.2.1 Consider an $n \times n$ matrix A and a scalar λ. Then λ is an eigenvalue of A if (and only if) $\det(\lambda I_n - A) = 0$

λ is an eigenvalue of A.

$$\Updownarrow$$

There is a nonzero vector $\vec{v}$ such that $A\vec{v} = \lambda\vec{v}$ or $(\lambda I_n - A)\vec{v} = \vec{0}$.

$$\Updownarrow$$

$ker(\lambda I_n - A) \neq \{\vec{0}\}$.

$$\Updownarrow$$

$\lambda I_n - A$ is not invertible.

$$\Updownarrow$$

$det(\lambda I_n - A) = 0$

EXAMPLE 1 Find the eigenvalues of the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}.$$

Solution

By Fact 7.2.1, we have to solve the equation det($\lambda I_2 - A$)=0:

$$det(\lambda I_2 - A) = det\left(\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}\right)$$

$$= det\begin{bmatrix} \lambda - 1 & -2 \\ -4 & \lambda - 3 \end{bmatrix}$$

$$= (\lambda - 1)(\lambda - 3) - 8$$

$$= \lambda^2 - 4\lambda - 5$$

$$= (\lambda - 5)(\lambda + 1) = 0$$

The matrix A have two eigenvalues 5 and -1.

EXAMPLE 2 Find the eigenvalues of

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 2 & 3 & 4 & 5 \\ 0 & 0 & 3 & 4 & 5 \\ 0 & 0 & 0 & 4 & 5 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix}.$$

Solution

Again, we have to solve the equation $\det(\lambda I_5 - A) = 0$:

$$det(\lambda I_5 - A) = \begin{bmatrix} \lambda-1 & -2 & -3 & -4 & -5 \\ 0 & \lambda-2 & -3 & -4 & -5 \\ 0 & 0 & \lambda-3 & -4 & -5 \\ 0 & 0 & 0 & \lambda-4 & -5 \\ 0 & 0 & 0 & 0 & \lambda-5 \end{bmatrix}$$

$$= (\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)(\lambda-5) = 0$$

There are five eigenvalues 1, 2, 3, 4, and 5 for matrix A.

Fact 7.2.2 The eigenvalues of a **triangular matrix** are its diagonal entries.

The eigenvalues of an $n \times n$ matrix A as zeros of the function

$$f_A(\lambda) = det(\lambda I_n - A).$$

EXAMPLE 3 Find $f_A(\lambda)$ for the 2×2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Solution

$$f_A(\lambda) = det(\lambda I_2 - A) = det \begin{bmatrix} \lambda - a & -b \\ -c & \lambda - d \end{bmatrix}$$

$$= \lambda^2 - (a + d)\lambda + (ad - bc)$$

The constant term is $det(A)$. Why? Because the constant term is $f_A(0)$=det($0I_2-A$)=det($-A$) =det(A).

Meanwhile, the coefficient of λ is the negative of the sum of the diagonal entries of A. Since the sum is important in many other contexts, we introduce a name for it.

Definition 7.2.3 Trace
The sum of the diagonal entries of an $n \times n$ matrix A is called the *trace* of A, denoted by tr(A).

Fact 7.2.4 If A is a 2×2, then

$$f_A(\lambda) = det(\lambda I_2 - A) = \lambda^2 - tr(A)\lambda + det(A)$$

For the matrix $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$, we have tr($A$)=4 and det($A$)=-5, so that

$$f_A(\lambda) = \lambda^2 - 4\lambda - 5.$$

What is the format of $f_A(\lambda)$ for an $n \times n$ matrix A?

$$f_A(\lambda) = \lambda^n - tr(A)\lambda^{n-1} + \text{(a polynomial of degree} \le (n-2)).$$

The constant term of this polynomial is $f_A(0) = det(-A) = (-1)^n det(A)$.

Fact 7.2.5 Characteristic polynomial
Consider an $n \times n$ matrix A. Then $f_A(\lambda) = det(\lambda I_n - A)$ is a polynomial of degree n of the form

$$f_A(\lambda) = \lambda^n - tr(A)\lambda^{n-1} + \cdots + (-1)^n det(A)$$

$f_A(\lambda)$ is called the *characteristic polynomial* of A

From elementary algebra, a polynomial of degree n has at most n zeros. If n is odd,

$$\lim_{\lambda \to \infty} f_A(\lambda) = \infty \text{ and } \lim_{\lambda \to -\infty} f_A(\lambda) = -\infty.$$

See Figure 1.

EXAMPLE 4 Find the eigenvalues of

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 2 & 3 & 4 & 5 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Solution

Since $f_A(\lambda) = (\lambda - 1)^3(\lambda - 2)^2$, the eigenvalues are 1 and 2. Since 1 is a root of multiplicity 3 of the characteristic polynomial, we say that the eigenvalue 1 has **algebraic multiplicity** 3. Likewise, the eigenvalue 2 has algebraic multiplicity 2.

Definition 7.2.6

Algebraic multiplicity of an eigenvalue We say that an eigenvalue λ_0 of a square matrix A has *algebraic multiplicity* k if

$$f_A(\lambda) = (\lambda - \lambda_0)^k g(\lambda)$$

for some polynomial $g(\lambda)$ with $g(\lambda_0) \neq 0$ (i.e.,if λ_0 is a root of multiplicity k of $f_A(\lambda)$).

EXAMPLE 5 Find the eigenvalues of

$$A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

with their algebraic multiplicities.

Solution

$$f_A(\lambda) = det \begin{bmatrix} \lambda - 2 & 1 & 1 \\ 1 & \lambda - 2 & 1 \\ 1 & 1 & \lambda - 2 \end{bmatrix}$$

$$= (\lambda - 2)^3 + 2 - 3(\lambda - 2) = (\lambda - 3)^2 \lambda$$

We found two distinct eigenvalues, 3 and 0, with algebraic multiplicities 2 and 1, respectively.

Fact 7.2.7 An $n \times n$ matrix has at most n eigenvalues, even if they are counted with their algebraic multiplicities.

If n is odd, then an $n \times n$ matrix has at least one eigenvalue.

EXAMPLE 6 Describe all possible cases for the number of real eigenvalues of a 3×3 matrix and their algebraic multiplicities. Give an example in each case and graph the characteristic polynomial.

Solution

Case 1: See Figure 3.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}, f_A(\lambda) = (\lambda - 1)(\lambda - 2)(\lambda - 3).$$

Case 2: See Figure 4.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}, f_A(\lambda) = (\lambda - 1)^2(\lambda - 2).$$

Case 3: See Figure 5.

$$A = I_3, f_A(\lambda) = (\lambda - 1)^3.$$

Case 4: See Figure 6.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, f_A(\lambda) = (\lambda - 1)(\lambda^2 + 1).$$

It is usually impossible to find the exact eigenvalue of a matrix. To find approximations for the eigenvalues, you could graph the characteristic polynomial. The graph may give you an idea of the number of eigenvalues and their approximate values. Numerical analysts tell us that this is not a very efficient way to go; other techniques are used in practice. (See Exercise 7.5.33 for an example; another approach uses QR factorization.)

Exercises 7.2: 3, 5, 9, 11, 18, 20, 25