3.4 COORDINATES

EXAMPLE 1

Let V be the plane in R^3 with equation $x_1+2x_2+3x_3=0$, a two-dimensional subspace of R^3. We can describe a vector in this plane by its spatial (3D)coordinates; for example, vector

$$\vec{x} = \begin{bmatrix} 5 \\ -1 \\ -1 \end{bmatrix}$$

is in plane V. However, it may be more convenient to introduce a plane coordinate system in V.

Consider any two vectors in plane V that aren't parallel, e.g.

$$\vec{v_1} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} \text{ and } \vec{v_2} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

See Figure 1, where we label the new axes c_1 and c_2, with the new coordinate grid defined by vectors $\vec{v_1}$ and $\vec{v_2}$.

Note that the c_1 - c_2 coordinates of vector $\vec{v_1}$ is $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and the coordinates of vector $\vec{v_2}$ is $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, respectively.

For a vector $\vec{x}$ in plane V, we can find the scalars c_1 and c_2 such that

$$\vec{x} = c_1\vec{v_1} + c_2\vec{v_2}.$$

For example, $\vec{x} = \begin{bmatrix} 5 \\ -1 \\ -1 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} + 2\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

Therefore, the $c_1 - c_2$ coordinates of $\vec{x}$ are

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

See Figure 3.

Let's denote the basis $\vec{v_1}$, $\vec{v_2}$ of V by B (Fraktur B). Then, the coordinate vector of $\vec{x}$ with respect to B is denoted by $\left[\ \vec{x}\ \right]_B$:

$$\text{If } \vec{x} = \begin{bmatrix} 5 \\ -1 \\ -1 \end{bmatrix}, \text{ then } \left[\ \vec{x}\ \right]_B = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

Definition 3.4.1
Coordinates in a subspace of R^n

Consider a basis B of a subspace V of R^n, consisting of vectors $\vec{v_1}$, $\vec{v_2}$, ..., $\vec{v_m}$. Any vector $\vec{x}$ in V can be written uniquely as

$$\vec{x}=c_1\vec{v_1}+c_2\vec{v_2}+...+c_m\vec{v_m}$$

The scalars c_1, c_1, ..., c_m are called the B-coordinates of $\vec{x}$, and the vector

$$\begin{bmatrix} c_1 \\ c_2 \\ ... \\ c_m \end{bmatrix}$$

is called the B-coordinate vector of $\vec{x}$, denoted by $\left[\ \vec{x}\ \right]_B$.

Note that

$$\vec{x} = S\left[\ \vec{x}\ \right]_B$$

where $S = \begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \dots & \vec{v}_m \\ | & | & & | \end{bmatrix}$, an n$\times m$ matrix.

EXAMPLE 2

Consider the basis B of R^2 consisting of vectors $\vec{v_1} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $\vec{v_2} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$

a. If $\vec{x} = \begin{bmatrix} 10 \\ 10 \end{bmatrix}$, find $\left[\vec{x} \right]_B$

b. If $\left[\vec{x} \right]_B = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, find $\vec{x}$

Solution

a. To find the coordinates of vector $\vec{x}$, we need to write $\vec{x}$ as a linear combination of the basis vectors:

$\vec{x} = c_1\vec{v_1} + c_2\vec{v_2}$, or $\begin{bmatrix} 10 \\ 10 \end{bmatrix} = c_1 \begin{bmatrix} 3 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 3 \end{bmatrix}$

Alternatively, we can solve the equation

$$\vec{x} = S\left[\vec{x} \right]_B = \begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix} \left[\vec{x} \right]_B$$

for $\left[\vec{x} \right]_B = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

$$\left[\ \vec{x}\ \right]_B = S^{-1}\vec{x} = \begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 10 \\ 10 \end{bmatrix}$$

$$= \frac{1}{10}\begin{bmatrix} 3 & 1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 10 \\ 10 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

b. By definition of coordinates, $\left[\ \vec{x}\ \right]_B = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ means that

$$\vec{x} = 2\vec{v_1} + (-1)\vec{v_2} = 2\begin{bmatrix} 3 \\ 1 \end{bmatrix} + (-1)\begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 7 \\ -1 \end{bmatrix}$$

Alternatively, use the formula

$$\vec{x} = S\left[\ \vec{x}\ \right]_B = \begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 7 \\ -1 \end{bmatrix}$$

EXAMPLE 3

Let L be the line in R^2 spanned by vector $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

Let T be the linear transformation from R^2 to R^2 that projects any vector orthogonally onto line L, as shown in Figure 5.

1. In $\vec{x}_1 - \vec{x}_2$ coordinate system (See Figure 5): Sec 2.2 (pp. 59).

2. In $c_1 - c_2$ coordinate system (See Figure 6): T transforms vector $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ into $\begin{bmatrix} c_1 \\ 0 \end{bmatrix}$.

   That is, T is given by the matrix $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, since $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} c_1 \\ 0 \end{bmatrix}$

The transforms from $\left[\vec{x} \right]_B$ into $\left[T(\vec{x}) \right]_B$ is called the B-matrix of T:

$$\left[T(\vec{x}) \right]_B = B \left[\vec{x} \right]_B$$

Definition 3.4.2
The B-matrix of a linear transformation
Consider a linear transformation T from R^n to R^n and a basis B of R^n. The $n \times n$ matrix B that transforms $\left[\ \vec{x}\ \right]_B$ into $\left[\ T(\vec{x})\ \right]_B$ is called the B-matrix of T:

$$\left[\ T(\vec{x})\ \right]_B = \text{B} \left[\ \vec{x}\ \right]_B$$

for all $\vec{x}$ in R^n.

Fact 3.4.3 The columns of the B-matrix of a linear transformation
Consider a linear transformation T from R^n to R^n and a basis B of R^n consisting of vectors $\vec{v_1}$, $\vec{v_2}$, ..., $\vec{v_n}$. Then, the B-matrix of T is

$$B = \left[\ \left[\ T(\vec{x_1})\ \right]_B \left[\ T(\vec{x_2})\ \right]_B \cdots \left[\ T(\vec{x_n})\ \right]_B\ \right]$$

That is, the columns of B are the B-coordinate vectors of T($\vec{v_1}$), T($\vec{v_2}$), ..., T($\vec{v_n}$).

EXAMPLE 4

Consider two perpendicular unit vectors $\vec{v_1}$ and $\vec{v_2}$ in R^3. Form the basis $\vec{v_1}$, $\vec{v_2}$, $\vec{v_3}$=$\vec{v_1} \times \vec{v_2}$ of R^3; let's denote this basis by B. Find the B-matrix B of the linear transformation T($\vec{x}$)=$\vec{v_1} \times \vec{x}$.

(see Exercise 2.1: 44 on pp. 49,
$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \times \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{bmatrix}$)

Solution

Use Fact 3.4.3 to construct B column by column:

$$B = \Big[\ \big[\ T(\vec{x_1}) \ \big]_B \big[\ T(\vec{x_2}) \ \big]_B \cdots \big[\ T(\vec{x_n}) \ \big]_B \ \Big]$$
$$= \Big[\ \big[\ \vec{v_1} \times \vec{v_1} \ \big]_B \big[\ \vec{v_1} \times \vec{v_2} \ \big]_B \big[\ \vec{v_1} \times \vec{v_3} \ \big]_B \ \Big]$$
$$= \Big[\ \big[\ \vec{0} \ \big]_B \big[\ \vec{v_3} \ \big]_B \big[\ -\vec{v_2} \ \big]_B \ \Big]$$
$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

EXAMPLE 5

Let T be the linear transformation from R^2 to R^2 that projects any vector orthogonally onto the line L spanned by $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$. In Example 3, we found that the matrix of T with respect to the basis B consisting of $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$ is

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

What is the relation ship between B and the standard matrix A of T (such that T($\vec{x}$)=A$\vec{x}$)?

Solution

Recall from Definition 3.4.1 that

$$\vec{x} = S \begin{bmatrix} \vec{x} \end{bmatrix}_B, \text{ where } S = \begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix}$$

and consider the following diagram: (Figure 7)

Note that T($\vec{x}$)=AS $\left[\ \vec{x}\ \right]_B$
and also T($\vec{x}$)=SB $\left[\ \vec{x}\ \right]_B$,
so that AS $\left[\ \vec{x}\ \right]_B$ =SB $\left[\ \vec{x}\ \right]_B$ for all $\vec{x}$.

Thus,

$$\text{AS=SB and A=SB}S^{-1}$$

Now we can find the standard matrix A of T:

A=SBS^{-1}
$= \begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \left(\frac{1}{10} \begin{bmatrix} 3 & 1 \\ -1 & 3 \end{bmatrix} \right)$
$= \begin{bmatrix} 0.9 & 0.3 \\ 0.3 & 0.1 \end{bmatrix}$

Alternatively, we could use Fact 2.2.5 to construct matrix A. The point here was to explore the relationship between matrices A and B.

Fact 3.4.4
Standard matrix versus B-matrix of a linear transformation
Consider a linear transformation T from R^n to R^n and a basis B of R^n consisting of vectors $\vec{v_1}$, $\vec{v_2}$, ..., $\vec{v_n}$. Let B be the B-matrix of T and let A be the standard matrix of T(such that T($\vec{x}$)=A$\vec{x}$). Then, $AS = SB$, $B = S^{-1}AS$, and $A = SBS^{-1}$, where

$$S = \begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \dots & \vec{v}_m \\ | & | & & | \end{bmatrix}$$

Definition 3.4.5 Similar matrices
Consider two n $\times$ n matrices A and B. We say that A is similar to B if there is an invertible matrix S such that

AS=SB, or B=S^{-1}AS

EXAMPLE 6

Is matrix $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$ similar to $B = \begin{bmatrix} 5 & 0 \\ 0 & -1 \end{bmatrix}$?

Solution

We are looking for a matrix S= $\begin{bmatrix} x & y \\ z & t \end{bmatrix}$ such that AS=SB, or

$$\begin{bmatrix} x+2z & y+2t \\ 4x+3z & 4y+3t \end{bmatrix} = \begin{bmatrix} 5x & -y \\ 5z & -t \end{bmatrix}.$$

These equations simplify to

$$z = 2x, t = -y,$$

so that any invertible matrix of the form

$$S = \begin{bmatrix} x & y \\ 2x & -y \end{bmatrix}$$

does the job. Note that det(S)=-3xy. Matrix S is invertible if det(S)≠0 (i.e.,if neither x nor y is zero).

EXAMPLE 7

Show that if matrix A is similar to B, then its power A^t is similar to B^t for all positive integers t. (That is, A^2 is similar to B^2, A^3 is similar to B^3, etc.)

Solution

We know that B=S^{-1}AS for some invertible matrix S. Now, B^t

$$= \underbrace{(S^{-1}AS)(S^{-1}AS)...(S^{-1}AS)(S^{-1}AS)}_{t-times}$$

$= S^{-1}A^tS$,

proving our claims. Note the cancellation of many terms of the form SS^{-1}.

Fact 3.4.6
Similarity is an equivalence relation

1. An $n \times n$ matrix A is similar to itself (Reflexivity).

2. If A is similar to B, then B is similar to A (Symmetry).

3. If A is similar to B and B is similar to C, then A is similar to C (Transitivity).

Proof
A is similar to B: $B = P^{-1}AP$
B is similar to C: $C = Q^{-1}BQ$, then

$$C = Q^{-1}BQ = Q^{-1}P^{-1}APQ = (PQ)^{-1}A(PQ)$$

that is, A is similar to C by matrix PQ.

Homework Exercise 3.4: 5, 6, 9, 10, 13, 14, 19, 31, 39