3.2 Subspaces of R^n Bases and Linear Independence

Definition. Subspaces of R^n
A subset W of R^n is called a subspace of R^n if it has the following properties:

(a). W contains the zero vector in R^n.
(b). W is closed under addition.
(c). W is closed under scalar multiplication.

Fact 3.2.2
If T is a linear transformation from R^n to R^m, then

$\diamond$ $ker(T)$ is a subspace of R^n
$\diamond$ $im(T)$ is a subspace of R^m

Example. *Is* $W = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in R^2 : x \geq 0, y \geq 0 \right\}$ *a subspace of* R^2*?*

See Figure 1, 2.

Example. *Is* $W = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in R^2 : xy \geq 0 \right\}$ *a subspace of* R^2*?*

See Figure 3, 4.

Example. *Show that the only subspaces of* R^2 *are:* $\{\vec{0}\}$*, any lines through the origin, and* R^2 *itself.*

Similarly, the only subspaces of R^3 are: $\{\vec{0}\}$, any lines through the origin, any planes through $\vec{0}$, and R^3 itself.

Solution

Suppose W is a subspace of R^2 that is neither the set $\{\vec{0}\}$ nor a line through the origin. We have to show $W = R^2$.

Pick a nonzero vector $\vec{v_1}$ in W. (We can find such a vector, since W is not $\{\vec{0}\}$.) The subspace W contains the line L spanned by $\vec{v_1}$, but W does not equal L. Therefore, we can find a vector $\vec{v_2}$ in W that is not on L (See Figure 5). Using a parallelogram, we can express any vector $\vec{v}$ in R^2 as a linear combination of $\vec{v_1}$ and $\vec{v_2}$. Therefore, $\vec{v}$ is contained in W (Since W is closed under linear combinations). This shows that $W = R^2$, as claimed.

A plane E in R^3 is usually described either by

$$x_1 + 2x_2 + 3x_3 = 0$$

or by giving E parametrically, as the span of two vectors, for example,

$$\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}.$$

In other words, E is described either as

$$ker \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$$

or

$$im \begin{bmatrix} 1 & 1 \\ 1 & -2 \\ -1 & 1 \end{bmatrix}$$

Similarly, a line L in R^3 may be described either parametrically, as the span of the vector

$$\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

or by two linear equations

$$\left| \begin{array}{c} x_1 - x_2 - x_3 = 0 \\ x_1 - 2x_2 + x_3 = 0 \end{array} \right|$$

Therfore

$$L = im \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = ker \begin{bmatrix} 1 & -1 & -1 \\ 1 & -2 & 1 \end{bmatrix}$$

A subspace of R^n is uaually presented either as the solution set of a homogeneous linear system (as a kernel) or as the span of some vectors (as an image).

Any subspace of R^n can be represented as the image of a matrix.

Bases and Linear Independence

Example. *Consider the matrix*

$$A = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 1 & 2 & 2 & 3 \\ 1 & 3 & 2 & 4 \end{bmatrix}$$

Find vectors $\vec{v_1}$, $\vec{v_2}$, $\cdots$, $\vec{v_m}$ *in* R^3 *that span the image of* A. *What is the* <u>*smallest number*</u> *of vectors needed to span the image of* A*?*

Solution

We know from Fact 3.1.3 that the image of A spanned by the columns of A,

$$\vec{v_1} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \vec{v_2} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \vec{v_3} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}, \vec{v_4} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$$

Figure 6 show that we need only $\vec{v_1}$ and $\vec{v_2}$ to span the image of A. Since $\vec{v_3} = \vec{v_2}$ and $\vec{v_4} = \vec{v_1} + \vec{v_2}$, the vectors $\vec{v_3}$ and $\vec{v_4}$ are redundant; that is, they are linear combinations of $\vec{v_1}$ and $\vec{v_2}$:

$$\begin{aligned} \text{im(A)} &= \text{span}(\vec{v_1}, \vec{v_2}, \vec{v_3}, \vec{v_4}) \\ &= \text{span}(\vec{v_1}, \vec{v_2}) \ . \end{aligned}$$

The image of A can be spanned by two vectors, but not by one vectors alone.

Definition. Linear independence; basis
Consider a sequence $\vec{v}_1, \ldots, \vec{v}_m$ of vectors in a subspace V of R^n.

The vectors $\vec{v}_1, \ldots, \vec{v}_m$ are called **linearly independent** if nono of them is a linear combination of the others.

We say that the vectors $\vec{v}_1, \ldots, \vec{v}_m$ form a **basis** of V if they span V and are linearly independent.

See last example. The vectors $\vec{v_1}$, $\vec{v_2}$, $\vec{v_3}$, $\vec{v_4}$ span

$$V = im(A)$$

but they are linearly dependent, because $\vec{v_4}=\vec{v_2}+\vec{v_3}$. Therefore, they do not form a basis of V. The vectors $\vec{v_1}$, $\vec{v_2}$, on the other hand, do span V and are linearly independent.

Definition. Linear relations
Consider the vectors $\vec{v}_1, \ldots, \vec{v}_m$ *in* R^n. *An equation of the form*

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_m\vec{v}_m = \vec{0}$$

is called a **(linear) relation** *among the vectors* $\vec{v}_i$. *There is always the trievial relation, with* $c_1 = c_2 = \cdots = c_m = 0$. *Nontrivial relations may or may not exist among the vectors* $\vec{v}_1, \ldots, \vec{v}_m$ *in* R^n.

Fact 3.2.5
The vectors $\vec{v}_1, \ldots, \vec{v}_m$ in R^n are linearly dependent if (and only if) there are nontrivial relations among them.

Proof
$\Rightarrow$ If one of the $\vec{v}_i$ s a linear combination of the others,

$$\vec{v}_i = c_1\vec{v}_1 + \cdots + c_{i-1}\vec{v}_{i-1} + c_{i+1}\vec{v}_{i+1} + \ldots + c_m\vec{v}_m$$

then we can find a nontrivial relation by subtracting $\vec{v}_i$ from both sides of the equations:

$$c_1\vec{v}_1 + \cdots + c_{i-1}\vec{v}_{i-1} - \vec{v}_i + c_{i+1}\vec{v}_{i+1} + \ldots + c_m\vec{v}_m = \vec{0}$$

$\Leftarrow$ Conversely, if there is a nontrivial relation

$$c_1\vec{v}_1 + \cdots + c_i\vec{v}_i + \ldots + c_m\vec{v}_m = \vec{0}$$

then we can solve for $\vec{v}_i$ and express $\vec{v}_i$ as a linear combination of the other vectors.

Example. *Determine whether the following vectors are linearly independent*

$$\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}, \begin{bmatrix} 6 \\ 7 \\ 8 \\ 9 \\ 10 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 5 \\ 7 \\ 11 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 9 \\ 16 \\ 25 \end{bmatrix}.$$

Solution

TO find the relations among these vectors, we have to solve the vector equation

$$c_1 \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix} + c_2 \begin{bmatrix} 6 \\ 7 \\ 8 \\ 9 \\ 10 \end{bmatrix} + c_3 \begin{bmatrix} 2 \\ 3 \\ 5 \\ 7 \\ 11 \end{bmatrix} + c_4 \begin{bmatrix} 1 \\ 4 \\ 9 \\ 16 \\ 25 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

or

$$\begin{bmatrix} 1 & 6 & 2 & 1 \\ 2 & 7 & 3 & 4 \\ 3 & 8 & 5 & 9 \\ 4 & 9 & 7 & 16 \\ 5 & 10 & 11 & 25 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

In other words, we have to find the $kernal$ of A. To do so, we compute $rref(A)$. Using technology, we find that

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This shows the kernel of A is $\{\vec{0}\}$, because there is a leading 1 in each column of $rref(A)$. There is only the trivial relation among the four vectors and they are therefore linearly independent.

Fact 3.2.6

The vectors $\vec{v}_1, \ldots, \vec{v}_m$ in R^n are linearly independent if (and only if)

$$ker \begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \ldots & \vec{v}_m \\ | & | & & | \end{bmatrix} = \{\vec{0}\}$$

or, equivalently, of

$$rank \begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \ldots & \vec{v}_m \\ | & | & & | \end{bmatrix} = m$$

This condition implies that $m \leq n$.

Fact 3.2.7
Consider the vectors $\vec{v}_1, \ldots, \vec{v}_m$ in a subspace V of R^n.
The vectors $\vec{v}_i$ are a basis of V if (and only if) every vector $\vec{v}$ in V can be expressed **uniquely** as a linear combination of the vectors $\vec{v}_i$.

Proof
$\Rightarrow$ Suppose vectors $\vec{v}_i$ are a basis of V, and consider a vector $\vec{v}$ in V. Since the basis vectors span V, the vector $\vec{v}$ can be written as a linear combination of the $\vec{v}_i$. We have to demonstrate that this representation is unique. If there are two representations:

$$\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_m\vec{v}_m$$
$$= d_1\vec{v}_1 + d_2\vec{v}_2 + \ldots + d_m\vec{v}_m$$

By subtraction, we find

$$\vec{0} = (c_1 - d_1)\vec{v}_1 + (c_2 - d_2)\vec{v}_2 + \ldots + (c_m - d_m)\vec{v}_m$$

Since the $\vec{v}_i$ are linearly independent, $c_i - d_i = 0$, or $c_i = d_i$, for all i.

$\Leftarrow$, suppose that each vector in V can be expressed uniquely as a linear combination of the vectors $\vec{v}_i$. Clearly, the $\vec{v}_i$. span V. The zero vector can be expressed uniquely as a linear combination of the $\vec{v}_i$, namely, as

$$\vec{0} = 0\vec{v}_1 + 0\vec{v}_2 + \ldots + 0\vec{v}_m$$

This means there is only the trivial relation among the $\vec{v}_i$: they are linearly independent.

See Figure 7. The vectors $\vec{v_1}$, $\vec{v_2}$, $\vec{v_3}$, $\vec{v_4}$ do not form a basis of E, since every vector in E can be expressed in more than one way as a linear combination of the $\vec{v}_i$. For example,

$$\vec{v}_4 = \vec{v}_1 + \vec{v}_2 + 0\vec{v}_3 + 0\vec{v}_4$$

but also

$$\vec{v}_4 = 0\vec{v}_1 + 0\vec{v}_2 + 0\vec{v}_3 + 1\vec{v}_4.$$

Homework 3.2: *3, 5, 9, 17, 18, 19, 29, 30, 39*