3.1 Image and Kernal of a Linear Transformation

Definition. Image
The image of a function consists of all the values the function takes in its codomain. If f is a function from X to Y, then

$$\begin{aligned} \text{image(f)} &= \{f(x)\colon\ x \in X\} \\ &= \{y \in Y\colon\ y = f(x), \text{ for some } x \in X\} \end{aligned}$$

Example. *See Figure 1.*

Example. *The image of*

$$f(x) = e^x$$

consists of all positive numbers.

Example. $b \in im(f), c \notin im(f)$ *See Figure 2.*

Example. $f(t) = \begin{bmatrix} cos(t) \\ sin(t) \end{bmatrix}$ *(See Figure 3.)*

Example. *If the function from* X *to* Y *is invertible, then* $image(f) = Y$. *For each* y *in* Y, *there is one (and only one)* x *in* X *such that* $y = f(x)$, *namely,* $x = f^{-1}(y)$.

Example. *Consider the linear transformation* T *from* R^3 *to* R^3 *that projects a vector orthogonally into the* $x_1 - x_2$*-plane, as illustrate in Figure 4. The image of* T *is the* $x_1 - x_2$*-plane in* R^3.

Example. *Describe the image of the linear transformation* T *from* R^2 *to* R^2 *given by the matrix*

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$$

Solution

$$T\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= x_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 6 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 3x_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$= (x_1 + 3x_2) \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

See Figure 5.

Example. *Describe the image of the linear transformation T from R^2 to R^3 given by the matrix*

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}$$

Solution

$$T \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

See Figure 6.

Definition. *Consider the vectors* $\vec{v}_1$, $\vec{v}_2$, . . . , $\vec{v}_n$ *in* R^m. *The set of all linear combinations of the vectors* $\vec{v}_1$, $\vec{v}_2$, . . . , $\vec{v}_n$ *is called their* **span***:*

span($\vec{v}_1$, $\vec{v}_2$, . . . , $\vec{v}_n$)
$=\{c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_n\vec{v}_n$: c_i *arbitrary scalars*$\}$

Fact The image of a linear transformation

$$T(\vec{x}) = A\vec{x}$$

is the span of the columns of A. We denote the image of T by $im(T)$ or $im(A)$.

Justification

$$T(\vec{x}) = A\vec{x} = \begin{bmatrix} | & & | \\ \vec{v_1} & \ldots & \vec{v_n} \\ | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$= x_1\vec{v_1} + x_2\vec{v_2} + \ldots + x_n\vec{v_n}.$$

Fact: Properties of the image

(a). The zero vector is contained in $im(T)$, i.e. $\vec{0} \in im(T)$.

(b). The image is closed under addition: If $\vec{v}_1$, $\vec{v}_2 \in im(T)$, then $\vec{v}_1 + \vec{v}_2 \in im(T)$.

(c). The image is closed under scalar multiplication: If $\vec{v} \in im(T)$, then $k\vec{v} \in im(T)$.

Verification

(a). $\vec{0} \in R^m$ since $A\vec{0} = \vec{0}$.

(b). Since $\vec{v_1}$ and $\vec{v_2} \in im(T)$, $\exists$ $\vec{w_1}$ and $\vec{w_2}$ st. $T(\vec{w_1}) = \vec{v_1}$ and $T(\vec{w_2}) = \vec{v_2}$. Then, $\vec{v_1} + \vec{v_2} = T(\vec{w_1}) + T(\vec{w_2}) = T(\vec{w_1} + \vec{w_2})$, so that $\vec{v_1} + \vec{v_2}$ is in the image as well.

(c). $\exists$ $\vec{w}$ st. $T(\vec{w}) = \vec{v}$. Then $k\vec{v} = kT(\vec{w}) = T(k\vec{w})$, so $k\vec{v}$ is in the image.

Example. *Consider an $n \times n$ matrix A. Show that $im(A^2)$ is contained in $im(A)$.*

Hint: To show $\vec{w}$ is also in $im(A)$, we need to find some vector $\vec{u}$ st. $\vec{w} = A\vec{u}$.

Solution

Consider a vector $\vec{w}$ in $im(A^2)$. There exists a vector $\vec{v}$ st. $\vec{w} = A^2\vec{v} = AA\vec{v} = A\vec{u}$ where $\vec{u} = A\vec{v}$.

Definition. *Kernel*

The kernel of a linear transformation $T(\vec{x}) = A\vec{x}$ is the set of all zeros of the transformation (i.e., the solutions of the equation $A\vec{x} = \vec{0}$. See Figure 9.

We denote the kernel of T by $ker(T)$ or $ker(A)$.

For a linear transformation T from R^n to R^m,

- $im(T)$ is a subset of the codomain R^m of T, and

- $ker(T)$ is a subset of the domain R^n of T.

Example. *Consider the orthogonal project onto the $x_1 - x_2-$plane, a linear transformation T from R^3 to R^3. See Figure 10.*

The kernel of T consists of all vectors whose orthogonal projection is $\vec{0}$. These are the vectors on the x_3-axis (the scalar multiples of $\vec{e}_3$).

Example. *Find the kernel of the linear transformation T from R^3 to R^2 given by*

$$T(\vec{x}) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}$$

Solution

We have to solve the linear system

$$T(\vec{x}) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \vec{x} = \vec{0}$$

$$rref \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 2 & 3 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right]$$

$$\left| \begin{array}{ccccc} x_1 & & - & x_3 & = 0 \\ & x_2 & + & 2x_3 & = 0 \end{array} \right|$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t \\ -2t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

The kernel is the line spanned by $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$.

Example. *Find the kernel of the linear transformation T from R^5 to R^4 given by the matrix*

$$A = \begin{bmatrix} 1 & 5 & 4 & 3 & 2 \\ 1 & 6 & 6 & 6 & 6 \\ 1 & 7 & 8 & 10 & 12 \\ 1 & 6 & 6 & 7 & 8 \end{bmatrix}$$

Solution We have to solve the linear system $\mathrm{T}(\vec{x}) = \mathrm{A}\vec{0} = \vec{0}$

$$\mathrm{rref(A)} = \begin{bmatrix} 1 & 0 & -6 & 0 & 6 \\ 0 & 1 & 2 & 0 & -2 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The kernel of T consists of the solutions of the system

$$\left| \begin{array}{ccccc} x_1 & & -6x_3 & & +6x_5 = 0 \\ & x_2 & +2x_3 & & -2x_5 = 0 \\ & & & x_4 & +2x_5 = 0 \end{array} \right|$$

The solution are the vectors

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 6s - 6t \\ -2s + 2t \\ s \\ -2t \\ t \end{bmatrix}$$

where s and t are arbitrary constants .

$$\text{ker(T)} = \left\{ \begin{bmatrix} 6s - 6t \\ -2s + 2t \\ s \\ -2t \\ t \end{bmatrix} : \text{ s , t arbitrary scalars} \right\}$$

We can write

$$\begin{bmatrix} 6s - 6t \\ -2s + 2t \\ s \\ -2t \\ t \end{bmatrix} = \text{s} \begin{bmatrix} 6 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \text{t} \begin{bmatrix} -6 \\ 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

This shows that

$$\ker(T) = \operatorname{span}\left(\begin{bmatrix} 6 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -6 \\ 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}\right)$$

Fact 3.1.6: Properties of the kernel

(a) The zero vector $\vec{0}$ in R_n in in ker(T).
(b) The kernel is closed under addition.
(c) The kernel is closed under scalar multiplication.

The verification is left as Exercise 49.

Fact 3.1.7
1. Consider an m*n matrix A then

$$\text{ker(A)} = \{\vec{0}\}$$

if (and only if) rank(A) $= n$.(This implies that $n \leq m$.)

Check exercise 2.4 (35)

2. For a square matrix A,

$$\text{ker(A)} = \{\vec{0}\}$$

if (and only if) A is invertible.

Summary
Let A be an n*n matrix . The following statements are equivalent (i.e.,they are either all true or all false):

1. A is invertible.

2. The linear system A$\vec{x} = \vec{b}$ has a unique solution $\vec{x}$, for all $\vec{b}$ in R^n. (def 2.3.1)

3. rref(A) $= I_n$. (fact 2.3.3)

4. rank(A) $= n$. (def 1.3.2)

5. im(A) $= R^n$. (ex 3.1.3b)

6. ker(A) $= \{\vec{0}\}$. (fact 3.1.7)

Homework 3.1: *5, 6, 7, 14, 15, 16, 31, 33, 42, 43*