Applied Linear Algebra

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Chapter 4
Linear Spaces

Chia-Hui Chang
Email: chia@csie.ncu.edu.tw
National Central University, Taiwan

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4.1 Introduction to Linear Systems

Definition 4.1.1
Linear spaces A linear space V is a set endowed with
(1) a rule for addition (if f and g are in V, then so is $f + g$) and
(2) a rule for scalar multiplication (if f is in V and k in R, then kf is in V)
such that these operations satisfy the following eight rules (for all f, g, h in V and all c, k in R):

1. $(f + g) + h = f + (g + h)$

2. $f + g = g + f$

3. There is a *neutral element* n in V such that $f + n = f$, for all f in V. This n is unique and denoted by 0.

4. For each f in V there is a g in V such that $f+g=0$. this g is unique and denoted by $(-f)$

5. $k(f+g)=kf+kg$

6. $(c+k)f=cf+kf$

7. $c(kf)=(ck)f$

8. $1f=f$

Linear Combination

We say that an element f of a linear space is a *linear combination* of the elements $f_1, f_2, \ldots, f_n$ if

$$f = c_1 f_1 + c_2 f_2 + \cdots + c_n f_n$$

for some scalars $c_1, c_2, \cdots, c_n$.

EXAMPLE 9

Let $A = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}$. Show that $A^2 = \begin{bmatrix} 2 & 3 \\ 6 & 11 \end{bmatrix}$ is a linear combination of A and I_2.

Solution

We have to find scalars c_1 and c_2 such that

$$A^2 = c_1 A + c_2 I_2,$$

or

$$A^2 = \begin{bmatrix} 2 & 3 \\ 6 & 11 \end{bmatrix} = c_1 \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix} + c_2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Definition 4.1.2 Subspaces

A subspace W of a linear space V is called a $subspace$ of V if

1. W contains the neutral element 0 of V
2. W is closed under addition (if f and g are in W, then so is $f + g$).
3. W is closed under scalar multiplication (if f is in W and k is a scalar, then kf is in W).

We can summarize parts (2) and (3) by saying that W is closed under linear combinations.

Definition 4.1.3
Span, linear independence, basis, coordinates

Consider the elements $f_1, f_2, \ldots, f_n$ of a linear space V.

1. We say that $f_1, f_2, \ldots, f_n$ *span* V if every f in V can be expressed as a linear combination of $f_1, f_2, \ldots, f_n$.

2. We say that $f_1, f_2, \ldots, f_n$ are (*linearly*) *independent* if the equation

$$c_1 f_1 + c_2 f_2 + \cdots + c_n f_n = 0$$

has only the trivial solution

$$c_1 = c_2 = \cdots = c_n = 0.$$

3. We say that elements $f_1, f_2, \ldots, f_n$ are a *basis* of V if they span V and are independent. This means that every f in V can be written uniquely as a linear combination

   $$f = c_1 f_1 + c_2 f_2 + \cdots + c_n f_n.$$

   The coefficients $c_1, c_2, \ldots, c_n$ are called the *coordinates* of f with respect to the basis $f_1, f_2, \ldots, f_n$.

Fact 4.1.4 Dimension

If a linear space V has a basis with n elements, then all other bases of V consist of n elements as well. We say that n is the *dimension* of V:

$$dim(V) = n.$$

Definition 4.1.6 Finite-dimensional linear spaces

A linear spaces V is called $finite-dimensional$ if it has a (finite) basis $f_1, f_2, \ldots, f_n$, so that we can define its dimension $dim(V) = n$. (See Definition 4.1.4.) Otherwise, the space is called $infinite-dimensional$.

Exercises 4.1: 3, 5, 7, 8, 17, 18, 20, 33, 35

EXAMPLE

In R^3, the prototype linear space, the neutral element is the zero vector, $\vec{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

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EXAMPLE 3

Let F(R,R) be the set of all functions from R to R (see Example 1), with the operations

$$(f+g)(x) = f(x) + g(x)$$

and

$$(kf)(x) = kf(x)$$

Then, F(R,R) is a linear space. The neutral element is the zero function, $f(x) = 0$ for all x.

EXAMPLE 11

The differentiable functions form a subspace W of $F(R,R)$.

EXAMPLE 12
Here are three more subspaces of F(R,R):

1. C^{∞}, the smooth functions, that is, functions we can differentiate as many times as we want. This subspace contains all polynomials, exponential functions, $sin(x)$, and $cos(x)$, for example.

2. P, the set of all polynomials.

3. P_n, the set of all polynomials of degree $\leq n$

EXAMPLE 11
The polynomials of degree$\leq$ 2, of the form $f(x) = a + bx + cx^2$, are a subspace W of the space F(R,R) of all functions from R to R.

EXAMPLE 16
Find a basis of P_2, the space of all polynomials of degree $\leq$ 2, and thus determine the dimension of P_2.

EXAMPLE 19
Let $f_1, f_2, \ldots, f_n$ be polynomials. Explain why these polynomials do not span the space P of all polynomials.

This implies that the space P of all polynomials does not have a finite basis $f_1, f_2, \ldots, f_n$.

EXAMPLE 4

If addition and scalar multiplication are given as in Definition 1.3.9, then $R^{2\times 2}$, the set of all 2×2 matrices, is a linear space. The neutral element is the zero matrix whose entries are all zero $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

EXAMPLE 13

Show that the matrices B that commute with $A = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}$ form a subspace of $R^{2\times 2}$.

EXAMPLE 14

Consider the set W of all noninvertible 2×2 matrices. Is W a subsequence of $R^{2\times 2}$?

EXAMPLE 15

Find a basis of $V = R^{2\times 2}$ and thus determine $dim(V)$.

EXAMPLE 17

Find a basis of the space V of all matrices B that commute with $A = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}$.

EXAMPLE 6
The linear equation in three unknowns,

$$ax + by + cz = d,$$

where a, b, c, and d are constants, from a linear space.

The neutral element is the equation $0 = 0$ (with $a = b = c = d = 0$).

Exercises 4.1: 3, 5, 7, 8, 17, 18, 20, 33, 35

4.2 LINEAR TRANSFORMATIONS AND ISOMORPHISMS

Definition 4.2.1

Linear transformation Consider two linear spaces V and W. A function T from V to W is called a linear transformation if:

$$T(f+g) = T(f) + T(g) \text{ and } T(kf) = kT(f)$$

for all elements f and g of V and for all scalar k.

Image, Kernel For a linear transformation T from V to W, we let

$$im(T) = \{T(f) : f \in V\}$$

and

$$ker(T) = \{f \in V : T(f) = 0\}$$

Note that $im(T)$ is a subspace of co-domain W and $ker(T)$ is a subspace of domain V.

Rank, Nullity

If the image of T is finite-dimensional, then $dim(imT)$ is called the rank of T, and if the kernel of T is finite-dimensional, then $dim(kerT)$ is the nullity of T.

If V is finite-dimensional, then the rank-nullity theorem holds (see fact 3.3.9):

$$\begin{aligned} \text{dim(V)} &= \text{rank(T)+nullity(T)} \\ &= \text{dim(imT)+dim(kerT)} \end{aligned}$$

EXAMPLE 4 Consider the transformation

$$T\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

from R^4 to $R^{2\times 2}$.
We are told that T is a linear transformation. Show that transformation T is invertible.

Solution
The most direct way to show that a function is invertible is to find its inverse. We can see that

$$T^{-1}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

The linear spaces R^4 and $R^{2\times 2}$ have essentially the same structure. We say that the linear spaces R^4 and $R^{2\times 2}$ are *isomorphic*.

Definition 4.2.2 Isomorphisms and isomorphic spaces

An invertible linear transformation is called an *isomorphism*. We say the linear space V and W are isomorphic if there is an isomorphism from V to W.

EXAMPLE 5 Show that the transformation

$$T(A) = S^{-1}AS \text{ from } R^{2\times2} \text{ to } R^{2\times2}$$

is an isomorphism, where $S = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

Solution
We need to show that T is a linear transformation, and that T is invertible.

Let's think about the linearity of T first:

$$T(M+N) = S^{-1}(M+N)S = S^{-1}(MS+NS)$$

$$= S^{-1}MS + S^{-1}NS$$

equals $T(M) + T(N) = S^{-1}MS + S^{-1}NS$ and

$$T(kA) = S^{-1}(kA)S = k(S^{-1}AS)$$

equals $kT(A) = k(S^{-1}AS)$.

The inverse transformation is

$$T^{-1}(B) = SBS^{-1}$$

Fact 4.2.3 Properties of isomorphisms

1. If T is an isomorphism, then so is T^{-1}

2. A linear transformation T from V to W is an isomorphism if (and only if)

$$ker(T) = \{0\}, im(T) = W$$

3. Consider an isomorphism T from V to W.If $f_1, f_2, ...f_n$
is a basis of V, then $T(f_1), T(f_2), ...T(f_n)$ is a basis of W.

4. If V and W are isomorphic and dim(V)=n, then dim(W)=n.

Proof

1. We must show that T^{-1} is linear. Consider two elements f and g of the codomain of T:

$$T^{-1}(f+g) = T^{-1}(TT^{-1}(f) + TT^{-1}(g))$$

$$= T^{-1}(T(T^{-1}(f) + T^{-1}(g)))$$

$$= T^{-1}(f) + T^{-1}(g)$$

In a similar way, you can show that $T^{-1}(kf) = kT^{-1}(f)$, for all f in the codomain of T and all scalars k.

2. $\Rightarrow$ To find the kernel of T, we have to solve the equation
$T(f) = 0$, Apply T^{-1} on both sides
$T^{-1}T(f) = T^{-1}(0), \rightarrow f = T^{-1}(0) = 0$
so that $ker(T) = 0$, as claimed.

Any g in W can be written as $g = T(T^{-1}(g))$, so that $im(T) = W$.

$\Leftarrow$ Suppose $ker(T) = \{0\}$ and $im(T) = W$. We have to show that T is invertible, i.e. the equation $T(f) = g$ has a unique solution f for any g in W.
There is at last one such solution, since $im(T) = W$. Prove by contradiction, consider two solutions f_1 and f_2:

$$T(f_1) = T(f_2) = g$$

$$0 = T(f_1) - T(f_2) = T(f_1 - f_2)$$

$$\Rightarrow f_1 - f_2 \in ker(T)$$

Since $ker(T) = \{0\}$, $f_1 - f_2 = 0, f_1 = f_2$

3. Span: For any g in W, there exists $T^{-1}(g)$ in V, we can write

$$T^{-1}(g) = c_1 f_1 + c_2 f_2 + \cdots + c_n f_n$$

because f_i span V. Applying T on both sides

$$g = c_1T(f_1) + c_2T(f_2) + \cdots + c_nT(f_n)$$

Independence: Consider a relation

$$c_1T(f_1) + c_2T(f_2) + \cdots + c_nT(f_n) = 0$$

or

$$T(c_1f_1 + c_2f_2 + \cdots + c_nf_n) = 0.$$

Since the ker(T) is $\{0\}$, we have

$$c_1f_1 + c_2f_2 + \cdots + c_nf_n = 0.$$

Since f_i are linear independent, the c_i are all zero.

4. Follows from part (c).

EXAMPLE 6 We are told that the transformation

$$B = T(A) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} A - A \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

from $R^{2\times 2}$ to $R^{2\times 2}$ is linear. Is T an isomorphism?

Solution We need to examine whether transformation T is invertible. First we try to solve the equation

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} A - A \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = B$$

for input A. However, the fact that matrix multiplication is non-commutative gets in the way, and we are unable to solve for A.

Instead, Consider the kernel of T:

$$T(A) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} A - A \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

or

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} A = A \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

We don't really need to find this kernal; we just want to know whether there are nonzero matrices in the kernel. Since I_2 and $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ is in the kernel, so that T is not isomophic.

Exercise 4.2: 5, 7, 9, 39

4.3 COORDINATES IN A LINEAR SPACE

By introducing coordinates, we can transform any n-dimensional linear space into R^n

4.3.1 Coordinates in a linear space

Consider a linear space V with a basis B consisting of $f_1, f_2, ...f_n$. Then any element f of V can be written uniquely as

$$\text{f} = c_1 f_1 + c_2 f_2 + \cdots + c_n f_n,$$

for some scalars $c_1, c_2, ..., c_n$. There scalars are called the B coordinates of f, and the vector

$$\begin{bmatrix} c_1 \\ c_2 \\ . \\ . \\ c_n \end{bmatrix}$$

is called the B-coordinate vector of f, denoted by $[f]_B$.

The B coordinate transformation $T(f) = [f]_B$ from V to R^n is an isomorphism (i.e., an invertible linear transformation). Thus, V is isomorphic to R^n; the linear spaces V and R^n have the same structure.

Example. *Choose a basis of P_2 and thus transform P_2 into R^n, for an appropriate n.*

Example. *Let V be the linear space of upper-triangular 2×2 matrices (that is, matrices of the form*

$$\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}.$$

Choose a basis of V and thus transform V into R^n, for an appropriate n.

Example. *Do the polynomials,* $f_1(x) = 1 + 2x + 3x^2$*,* $f_2(x) = 4 + 5x + 6x^2$*,* $f_3(x) = 7 + 8x + 10x^2$ *from a basis of* P_2*?*

Solution

Since P_2 is isomorphic to R^3, we can use a coordinate transformation to make this into a problem concerning R^3. The three given polynomials form a basis of P_2 if the coordinate vectors

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$$

form a basis of R^3.

Fact Two bases of a linear space consists of the same number of elements.

Proof Suppose two bases of a linear space V are given: basis $\coprod$, consisting of $f_1, f_2, \ldots, f_n$ and basis $\Im$ with m elements. We need to show that $m = n$.

Consider the vectors $[f_1]_{\Im}, [f_2]_{\Im}, \ldots, [f_n]_{\Im}$, these n vectors form a basis of R^m, since the $\Im$-coordinate transformation is an isomorphism from V to R^m.

Since all bases of R^m consist of m elements, we have $m = n$, as claimed.

Example. *Consider the linear transformation*

$$T\ (f) = f' + f'' \text{ form } P_2 \text{ to } P_2.$$

Since P_2 is isomorphic to R^3, this is essentially a linear transformation from R^3 to R^3, given by a 3×3 matrix B. Let's see how we can find this matrix.

Solution

We can write transformation T more explicitly as

$$\begin{aligned} T\ (a + bx + cx^2) &= (b + 2cx) + 2c \\ &= (b + 2c) + 2cx. \end{aligned}$$

Next let's write the input and the output of T in coordinates with respect to the standard basis B of P_2 consisting of $1, x, x^2$:

$$a + bx + cx^2 \longrightarrow (b + 2c) + 2cx$$

See Figure 1

Written in B coordinates, transformation T takes $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ into $\begin{bmatrix} b+2c \\ 2c \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

The matrix $B = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$ is called the matrix of T. It describes the transformation T if input and output are written in B coordinates.

Let us summarize our work in a diagram:

See Figure 2

Definition 4.3.2 B-Matrix of a linear transformation
Consider a linear transformation T from V to V, where V is an n-dimensional linear space. Let B be a basis of V. Then, there is an $n \times n$ matrix B that transform [f]$_B$ into $[T(f)]_B$, called the B-matrix of T.

$$[T(f)]_B = B[f]_B$$

Fact 4.3.3 The columns of the B-matrix of a linear transformation

Consider a linear transformation T from V to V, and let B be the matrix of T with respect to a basis$B$ of V consisting of $f_1, \ldots, f_n$.
Then

$$B = [[T(f_1)] \cdots [T(f_n)]] \, .$$

That is, the columns of B are the B-coordinate vectors of the transformation of the basis elements.

Proof

If

$$f = c_1 f_1 + c_2 f_2 + \cdots + c_n f_n,$$

then

$$T(f) = c_1 T(f_1) + c_2 T(f_2) + \cdots + c_n T(f_n),$$

and

$$[T(f)]_B = c_1 [T(f_1)]_B + c_2 [T(f_2)]_B + \cdots + c_n [T(f_n)]_B$$

$$= \left[\begin{array}{ccc} [T(f_1)]_B & \cdots & [T(f_n)]_B \end{array} \right] \left[\begin{array}{c} c_1 \\ \vdots \\ c_n \end{array} \right]$$

$$= \left[\begin{array}{ccc} [T(f_1)]_B & \cdots & [T(f_n)]_B \end{array} \right] [f]_B$$

Example. *Use Fact 4.3.3 to find the matrix B of the linear transformation*

$$T(f) = f' + f'' \text{ from } P_2 \text{ to } P_2$$

with respect to the standard basis B (See Example 4.)

Solution

$$B = \begin{bmatrix} [T(1)]_B & [T(x)]_B & [T(x^2)]_B \end{bmatrix}$$

$$B = \begin{bmatrix} [0]_B & [1]_B & [2+2x]_B \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Example. *Consider the function*

$$T(M) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} M - M \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

from $R^{2\times 2}$ *to* $R^{2\times 2}$. *We are told that* T *is a linear transformation.*

1. *Find the matrix* B *of* T *with respect to the standard basis* B *of* $R^{2\times 2}$
   (Hint: use column by column or definition)

2. *Find image and kernel of B.*

3. *Find image and kernel of T.*

4. *Find rank and nullity of transformation T.*

Solution

a. Use definition

$$T(M) = T\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} c & d \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ a & c \end{bmatrix} = \begin{bmatrix} c & d-a \\ 0 & -c \end{bmatrix}$$

Now we write input and output in B-coordinate:

See Figure 3

We can see that

$$B = \begin{bmatrix} 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

b. To find the kernel and image of matrix B, we compute rref(B) first:

$$rref(B) = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Therefore, $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ is a basis of ker(B)

and $\begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}$ is a basis of im(B).

c. To find image of kernel of T, we need to transform the vectors back into $R^{2\times 2}$:

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is a basis of ker(B)

and $\begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ is a basis of im(B).

d.

$$rank(T) = dim(imT) = 2$$

and

$$nullity(T) = dim(kerT) = 2.$$

Fact 4.3.4 The matrices of T with respect to different bases
Suppose that $\Im$ and B are two bases of a linear space V and that This a linear transformation from V to V.

1. There is an invertible matrix S such that $[f]_{\Im} = S[f]_B$ for all f in V.

2. Let A and B be the $\Im$ and the B-matrix of T, respectively. Then matrix A is $similar$ to B. In fact, B = $S^{-1}AS$ for the matrix S from part(a).

Proof
a. Suppose basis B consists of $f_1, f_2, \ldots, f_n$. If

$$f = c_1 f_1 + c_2 f_2 + \cdots + c_n f_n,$$

then

$$[f]_{\Im} = [c_1 f_1 + c_2 f_2 + \cdots + c_n f_n]_{\Im}$$

$$= c_1[f_1]_{\mathfrak{I}} + c_2[f_2]_{\mathfrak{I}} + \cdots + c_n[f_n]_{\mathfrak{I}}$$

$$= \begin{bmatrix} [f_1]_{\mathfrak{I}} & [f_2]_{\mathfrak{I}} & \cdots & [f_n]_{\mathfrak{I}} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \cdots \\ c_n \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} [f_1]_{\mathfrak{I}} & [f_2]_{\mathfrak{I}} & \cdots & [f_n]_{\mathfrak{I}} \end{bmatrix}}_{S} [f]_B$$

b. Consider the following diagram:

See Figure 4.

Performing a "diagram chase," we see that

$$AS = SB, \text{ or } B = S^{-1}AS.$$

See Figure 5.

Example. *Let V be the linear space spanned by functions e^x and e^{-x}. Consider the linear transformation D(f) = f' from V to V:*

1. *Find the matrix A of D with respect to basis B consisting of e^x and e^{-x}.*

2. *Find the matrix B of D with respect to basis B consisting of $(\frac{1}{2}(e^x+e^{-x}))$ and $(\frac{1}{2}(e^x-e^{-x}))$. (These two functions are called the $hypeerbolic$ cosine, cosh(x), and the $hypeerbolic$ sine, sinh(x), respectively.)*

3. *Using the proof of Fact 4.3.4 as a guide, construct a matrix S such that B $= S^{-1}AS$, showing that matrix A is similar to B.*

Exercise 4.3: 3, 7, 9, 13, 21, 34, 35, 37

Example Let V be the linear space of all functions of the form $f(x) = a\cos(x) + b\sin(x)$, a subspace of C^∞. Consider the transformation

$$T(f) = f'' - 2f' - 3f$$

from V to V.

1. Find the matrix B of T with respect to the basis B consisting of functions cos(x) and sin(x).

2. Is T an isomorphism?

3. How many solutions f in V does the differential equation

   $$f''(x) - 2f'(x) - 3f(x) = cos(x)$$

   have?