Applied Linear Algebra

OTTO BRETSCHER

http://www.prenhall.com/bretscher

Chapter 1
Linear Equations

Chia-Hui Chang
Email: chia@csie.ncu.edu.tw
National Central University, Taiwan

September 27, 2004

1.1 Introduction to Linear Systems

$$\left|\begin{array}{rrrcr} x & +2y & +3z & = & 39 \\ x & +3y & +2z & = & 34 \\ 3x & +2y & +z & = & 26 \end{array}\right| \Longrightarrow \left|\begin{array}{ccccl} x & & & = & \dots \\ & y & & = & \dots \\ & & z & = & \dots \end{array}\right|$$

$$\longrightarrow \left|\begin{array}{rrrcr} x & +2y & +3z & = & 39 \\ x & +3y & +2z & = & 34 \\ 3x & +2y & +z & = & 26 \end{array}\right| \begin{array}{l} \\ -(I) \\ \\ \end{array}$$

$$\longrightarrow \left|\begin{array}{rrrcr} x & +2y & +3z & = & 39 \\ & y & -z & = & -5 \\ 3x & +2y & +z & = & 26 \end{array}\right| \begin{array}{l} \\ \\ -3(I) \end{array}$$

$$\longrightarrow \left|\begin{array}{rrrcr} x & +2y & +3z & = & 39 \\ & y & -z & = & -5 \\ & -4y & -8z & = & -91 \end{array}\right| \begin{array}{l} -2(II) \\ \\ +4(II) \end{array}$$

$$\longrightarrow \left|\begin{array}{rrrcr} x & & +5z & = & 49 \\ & y & -z & = & -5 \\ & & -12z & = & -111 \end{array}\right| \begin{array}{l} \\ \\ \div(-12) \end{array}$$

$$\longrightarrow \left|\begin{array}{rrrcr} x & & +5z & = & 49 \\ & y & -z & = & -5 \\ & & z & = & 9.25 \end{array}\right| \begin{array}{l} -5(III) \\ +(III) \\ \\ \end{array}$$

$$\longrightarrow \left|\begin{array}{rrrcr} x & & & = & 2.75 \\ & y & & = & 4.25 \\ & & z & = & 9.25 \end{array}\right|$$

Geometric Interpretation

See Figure 1-3

Example. A System with Infinitely Many Solutions

$$\left|\begin{array}{l} 2x+4y+6z=0 \\ 4x+5y+6z=3 \\ 7x+8y+9z=6 \end{array}\right| \begin{array}{c} \div 2 \\ \longrightarrow \\ \end{array} \left|\begin{array}{rl} x & +2y+3z=0 \\ 4x & +5y+6z=3 \\ 7x & +8y+9z=6 \end{array}\right| \begin{array}{c} \longrightarrow \\ -4(I) \\ -7(I) \end{array}$$

$$\left|\begin{array}{r} x+2y+3z=0 \\ -3y-6z=3 \\ -6y-12z=6 \end{array}\right| \begin{array}{c} \longrightarrow \\ \div(-3) \\ \end{array} \left|\begin{array}{rrr} x+2y & +3z= & 0 \\ y & +2z= & -1 \\ -6y & -12z= & 6 \end{array}\right| \begin{array}{c} -2(II) \\ \longrightarrow \\ +6(II) \end{array}$$

$$\left|\begin{array}{lccr} x & -z & = & 2 \\ & y+2z & = & -1 \\ & 0 & = & 0 \end{array}\right| \longrightarrow \left|\begin{array}{lccr} x & -z & = & 2 \\ & y+2z & = & -1 \end{array}\right|$$

Example. A System without Solutions

$$\left|\begin{array}{rl} x+2y+3z & =0 \\ 4x+5y+6z & =3 \\ 7x+8y+9z & =0 \end{array}\right| \longrightarrow \left|\begin{array}{lrcr} x & -z & = & 2 \\ & y+2z & = & -1 \\ & 0 & = & -6 \end{array}\right|$$

1.2 Matrices and Gauss-Jordan Elimination

$$\left|\begin{array}{rcrcrcr} 2x & + & 8y & + & 4z & = & 2 \\ 2x & + & 5y & + & z & = & 5 \\ 4x & + & 10y & - & z & = & 1 \end{array}\right|$$

$$\text{Matrix} \rightarrow \begin{bmatrix} 2 & 8 & 4 & 2 \\ 2 & 5 & 1 & 5 \\ 4 & 10 & -1 & 1 \end{bmatrix}$$

Definition. *Coefficient matrix*

$$\begin{bmatrix} 2 & 8 & 4 \\ 2 & 5 & 1 \\ 4 & 10 & -1 \end{bmatrix}$$

Definition. *Augmented matrix*

$$\left[\begin{array}{rrrcr} 2 & 8 & 4 & \vdots & 2 \\ 2 & 5 & 1 & \vdots & 5 \\ 4 & 10 & -1 & \vdots & 1 \end{array}\right]$$

Definition. *Entry, Row, Column,*

Definition. *A matrix with only one column is called a* **column vector***, or simply a* **vector***. The entries of a vector are called its components. The set of all column vectors with n components is denoted by R^n.*

Example. $\begin{bmatrix} 1 \\ 2 \\ 9 \\ 1 \end{bmatrix}$ *is a (column) vector in R^4.*

Example. $\begin{bmatrix} 1 & 5 & 5 & 3 & 7 \end{bmatrix}$ *is a row vector in R^5.*

$$\left[\begin{array}{rrrcr} 2 & 8 & 4 & \vdots & 2 \\ 2 & 5 & 1 & \vdots & 5 \\ 4 & 10 & -1 & \vdots & 1 \end{array}\right] \begin{array}{l} \div 2 \\ \\ \\ \end{array}$$

$$\downarrow$$

$$\left[\begin{array}{rrrcr} 1 & 4 & 2 & \vdots & 1 \\ 2 & 5 & 1 & \vdots & 5 \\ 4 & 10 & -1 & \vdots & 1 \end{array}\right] \begin{array}{l} \\ -2(I) \\ -4(I) \end{array}$$

$$\downarrow$$

$$\left[\begin{array}{rrrcr} 1 & 4 & 2 & \vdots & 1 \\ 0 & -3 & -3 & \vdots & 3 \\ 0 & -6 & -9 & \vdots & -3 \end{array}\right] \begin{array}{l} \\ \div(-3) \\ \\ \end{array}$$

$$\downarrow$$

$$\left[\begin{array}{rrrcr} 1 & 4 & 2 & \vdots & 1 \\ 0 & 1 & 1 & \vdots & -1 \\ 0 & -6 & -9 & \vdots & -3 \end{array}\right] \begin{array}{l} -4(II) \\ \\ +6(II) \end{array}$$

$$\downarrow$$

$$\left[\begin{array}{ccccc} 1 & 0 & -2 & \vdots & 5 \\ 0 & 1 & 1 & \vdots & -1 \\ 0 & 0 & -3 & \vdots & -9 \end{array}\right] \begin{array}{l} \\ \\ \div(-3) \end{array}$$

$$\downarrow$$

$$\left[\begin{array}{ccccc} 1 & 0 & -2 & \vdots & 5 \\ 0 & 1 & 1 & \vdots & -1 \\ 0 & 0 & 1 & \vdots & 3 \end{array}\right] \begin{array}{r} +2(III) \\ -(III) \\ \\ \end{array}$$

$$\downarrow$$

$$\left[\begin{array}{ccccc} 1 & 0 & 0 & \vdots & 11 \\ 0 & 1 & 0 & \vdots & -4 \\ 0 & 0 & 1 & \vdots & 3 \end{array}\right]$$

The solution is represented as a vector:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 11 \\ -4 \\ 3 \end{bmatrix}$$

Gauss-Jordan Elimination

Step 0: row index=0, column index=0.

Step 1: If the cursor entry is 0, swap the cursor row with some row below to make the cursor entry nonzero.

Step 2: Divide the cursor row by the cursor entry to make the cursor entry equal to 1.

Step 3: Eliminate all other entries in the cursor column by subtracting suitable multiples of the cursor row from the other rows.

Step 4: Move the cursor down diagonally (i.e., down one row and over one column). If the new cursor entry and all entries below are zero, move the cursor to the next column (remaining in the same row). Repeat this step if necessary. Then return to step 1.

Definition. *We say that the matrix is in reduced* **row-echelon form**, *or* **rref** *for short. We write*

$$E\text{=}rref(M)$$

Definition. *A matrix is in* **reduced row-echelon form** *if it satisfies all of the following conditions:*

1. *If a row has nonzero entries, then the first nonzero entry is 1, called the* **leading** *1 in this row.*

2. *If a column contains a leading 1, then all other entries in that column are zero.*

3. *If a row contains a leading 1, then each row above contains a leading 1 further to the left.*

The method of solving a linear system by Gauss-Jordan elimination is called an **algorithm**.

An algorithm can be defined as "a finite procedure, written in a fixed symbolic vocabulary, governed by precise instructions, moving in discrete steps, 1, 2, 3, . . . , whose execution requires no insight, cleverness, intuition, intelligence, or perspicuity, and that sooner or later comes to an end."

Example.

$$\left|\begin{array}{ccccccccccc} & & & & x_3 & - & x_4 & - & x_5 & = & 4 \\ 2x_1 & + & 4x_2 & + & 2x_3 & + & 4x_4 & + & 2x_5 & = & 4 \\ 2x_1 & + & 4x_2 & + & 3x_3 & + & 3x_4 & + & 3x_5 & = & 4 \\ 3x_1 & + & 6x_2 & + & 6x_3 & + & 3x_4 & + & 6x_5 & = & 6 \end{array}\right|$$

Homework. *Exercises 1.2: 10, 11, 20, 34, 35*

1.3 On the Solutions of Linear Systems

A linear system has either

- **no solution** (*inconsistent*) iff its **rref** contains a row of the form

  $[\ 0\ 0\ 0\ \ldots\ 0\ \vdots 1]$

  representing the equation 0=1.

- **exactly one solution** if the system is consistent and all variables are leading, or

- **infinitely many solutions** if the system is consistent and there are nonleading variables.

Example. *The reduced row-echelon forms of the augmented matrices of three systems are given. How many solutions are there in each case?*

a. $\left[\begin{array}{ccc:c} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$

b. $\left[\begin{array}{ccc:c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right]$

c. $\left[\begin{array}{ccc:c} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$

Definition. *The* **rank** *of a matrix A is the number of leading 1's in rref(A).*

Example. $rank \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = 2$, *since*

$$rref \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Example. *Consider a system of m linear equations with n unkowns. Its coefficient matrix A has the size $m \times n$.*

1. $rank(A) \leq m$ and $rank(A) \leq n$.

2. If rank(A)=m, then the system is consistent.

3. If rank(A)=n, then the system has at most one solution.

4. If rank(A)$< n$, then the system has either infinitely many solutions, or none.

Example. *Consider a linear system with fewer equations than unknowns. How many solutions could this system have?*

Solution

Example. *Consider a linear system of n equations with n unknowns. When does this system have a unique solution?*

Solution

The **Vector Form** and the **Matrix Form** of a Linear System

$$\left| \begin{array}{rcrcl} 3x & + & y & = & 7 \\ x & + & 2y & = & 4 \end{array} \right|$$

We can write the system as:

$$\begin{bmatrix} 3x & + & y \\ x & + & 2y \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \end{bmatrix}$$

or

$$\begin{bmatrix} 3x \\ x \end{bmatrix} + \begin{bmatrix} y \\ 2y \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \end{bmatrix}$$

or

$$x \begin{bmatrix} 3 \\ 1 \end{bmatrix} + y \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \end{bmatrix}$$

See Figure 4 (pp.28).

Consider the general linear system

$$\left| \begin{array}{lll} a_{11}x_1 + a_{12}x_2 + \cdots + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + & a_{2n}x_n & = & b_2 \\ \vdots & \vdots & & \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + & a_{mn}x_n & = & b_m \end{array} \right|$$

We can write

$$\begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + & a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + & a_{2n}x_n \\ \vdots & \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + & a_{mn}x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

or

$$x_1 \underbrace{\begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}}_{\uparrow \atop \vec{v}_1} + x_2 \underbrace{\begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}}_{\uparrow \atop \vec{v}_2} + \cdots + x_n \underbrace{\begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}}_{\uparrow \atop \vec{v}_n} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}}_{\uparrow \atop \vec{b}}$$

Definition. Linear Combinations
A vector $\vec{b}$ in R^m is called a linear combination of the vectors $\vec{v}_1$, $\vec{v}_2$, . . . , $\vec{v}_n$ in R^m if there are scalars $x_1, x_2, \ldots, x_n$ such that

$$\vec{b} = x_1\vec{v}_1 + x_2\vec{v}_2 + \ldots + x_n\vec{v}_n$$

Definition. The product $A\vec{x}$
If the column vectors of an $m \times n$ matrix A are $\vec{v}_1$, $\vec{v}_2$, . . . , $\vec{v}_n$, and $\vec{x}$ is a vector in R^n, then the **product** *$A\vec{x}$ is defined as*

$$A\vec{x} = \begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$= x_1\vec{v}_1 + x_2\vec{v}_2 + \ldots + x_n\vec{v}_n.$$

In words, $A\vec{x}$ is the linear combination of the columns of A with the components of $\vec{x}$ as coefficients.

Example. $\begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$

$$= 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} 0 \\ 2 \end{bmatrix} + 2 \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 11 \end{bmatrix}$$

Example. $D = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ *and* $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$

find $D\vec{x}$

$$D\vec{x} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

$$= x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Example. *Represent the system in matrix form* $A\vec{x} = \vec{b}$.

$$\left| \begin{array}{l} 2x_1 - 3x_2 + 5x_3 = 7 \\ 9x_1 + 4x_2 - 6x_3 = 8 \end{array} \right|$$

$$\begin{bmatrix} 2 & -3 & 5 \\ 9 & 4 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 8 \end{bmatrix}$$

For an $m \times n$ matrix A, two vectors $\vec{x}$ and $\vec{y}$ in R^n, and a scalar k,

1. $A(\vec{x} + \vec{y}) = A\vec{x} + A\vec{y}$

2. $A(k\vec{x}) = k(A\vec{x})$

If $\vec{x}$ is a vector in R^n and A is an $m \times n$ matrix with row vectors $\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_m$, then

$$A\vec{x} = \begin{bmatrix} - & \vec{w}_1 & - \\ - & \vec{w}_2 & - \\ & \vdots & \\ - & \vec{w}_m & - \end{bmatrix} \vec{x} = \begin{bmatrix} \vec{w}_1 \cdot \vec{x} \\ \vec{w}_2 \cdot \vec{x} \\ \vdots \\ \vec{w}_m \cdot \vec{x} \end{bmatrix}$$

(That is, the ith component of $A\vec{x}$ is the dot product of $\vec{w}_i$ and $\vec{x}$.)

Example. $\begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$

$$= \begin{bmatrix} 1 \cdot 3 + 0 \cdot 1 + & (-1) \cdot 2 \\ 1 \cdot 3 + 2 \cdot 1 + & 3 \cdot 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 11 \end{bmatrix}$$

Example. $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 7 \\ 11 \end{bmatrix}$

$$= 7 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + 11 \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 29 \\ 65 \end{bmatrix}$$

or

$$= \begin{bmatrix} 1 \cdot 7 + 2 \cdot 11 \\ 3 \cdot 7 + 4 \cdot 11 \end{bmatrix} = \begin{bmatrix} 29 \\ 65 \end{bmatrix}$$

Definition. *The* **summation** *of two matrices of the same size is defined entry by entry.*

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} + \begin{bmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & & \vdots \\ b_{m1} & \cdots & b_{mn} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}+b_{11} & \cdots & a_{1n}+b_{1n} \\ \vdots & & \vdots \\ a_{m1}+b_{m1} & \cdots & a_{mn}+b_{mn} \end{bmatrix}$$

Definition. *The* **product** *of a scalar k with an $m \times n$ matrix is defined entry by entry.*

$$k \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} = \begin{bmatrix} ka_{11} & \cdots & ka_{1n} \\ \vdots & & \vdots \\ ka_{m1} & \cdots & ka_{mn} \end{bmatrix}$$

Example. $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} + \begin{bmatrix} 7 & 3 & 1 \\ 5 & 3 & -1 \end{bmatrix} = \begin{bmatrix} 8 & 5 & 4 \\ 9 & 8 & 5 \end{bmatrix}$

Example. $3 \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 6 & 3 \\ -3 & 9 \end{bmatrix}$

Example. *Consider an $m \times n$ matrix A with rank$(A) < m$. Find a vector $\vec{b}$ in R^m such that the system $A\vec{x} = \vec{b}$ is inconsistent.*

Solution

Let $E = rref(A)$. We can find a vector $\vec{c}$ in R^m such that the system $E\vec{x} = \vec{c}$ is inconsistent: Any vector whose last component is nonzero will do.

The key idea is to work backward through Gauss-Jordan elimination. For example, let

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 2 & 4 \\ 0 & 3 & 6 \\ 1 & 4 & 8 \end{bmatrix}, \text{ with } E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } \vec{c} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$
\begin{array}{cc}
A = \left[\begin{array}{ccc} 0 & 1 & 2 \\ 0 & 2 & 4 \\ 0 & 3 & 6 \\ 1 & 4 & 8 \end{array}\right] &
[A \vdots \vec{b}] = \left[\begin{array}{ccccc} 0 & 1 & 2 & : & 2 \\ 0 & 2 & 4 & : & 2 \\ 0 & 3 & 6 & : & 4 \\ 1 & 4 & 8 & : & 5 \end{array}\right] \\
\downarrow & \uparrow \\
\left[\begin{array}{ccc} 1 & 4 & 8 \\ 0 & 2 & 4 \\ 0 & 3 & 6 \\ 0 & 1 & 2 \end{array}\right]
\begin{array}{l} \\ \div(2) \\ \\ \\ \end{array} &
\left[\begin{array}{ccccc} 1 & 4 & 8 & : & 5 \\ 0 & 2 & 4 & : & 2 \\ 0 & 3 & 6 & : & 4 \\ 0 & 1 & 2 & : & 2 \end{array}\right] \\
\downarrow & \uparrow \\
\left[\begin{array}{ccc} 1 & 4 & 8 \\ 0 & 1 & 2 \\ 0 & 3 & 6 \\ 0 & 1 & 2 \end{array}\right]
\begin{array}{r} -4(II) \\ \\ -3(II) \\ -(II) \end{array} &
\left[\begin{array}{ccccc} 1 & 4 & 8 & : & 5 \\ 0 & 1 & 2 & : & 1 \\ 0 & 3 & 6 & : & 4 \\ 0 & 1 & 2 & : & 2 \end{array}\right]
\begin{array}{l} \\ \cdot(2) \\ \\ \\ \end{array} \\
\downarrow & \uparrow \\
E = \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] &
[E \vdots \vec{c}] = \left[\begin{array}{ccccc} 1 & 0 & 0 & : & 1 \\ 0 & 1 & 2 & : & 1 \\ 0 & 0 & 0 & : & 1 \\ 0 & 0 & 0 & : & 1 \end{array}\right]
\begin{array}{r} +4(II) \\ \\ +3(II) \\ +(II) \end{array}
\end{array}
$$

Homework. *Exercises 1.3: 1, 5, 8, 9, 19*

Write the pseudo code for Gauss-Jordan Elimination.