Chapter 7

One-dimensional Search Methods

Newton's Method

- By Taylor's expansion

$$g(x) = f(x^{(k)}) + f'(x^{(k)})(x - x^{(k)}) + \frac{1}{2} f''(x^{(k)})(x - x^{(k)})^2$$

- By the first-order necessary condition

$$g'(x) = 0 = f'(x^{(k)}) + f''(x^{(k)})(x - x^{(k)})$$ extreme

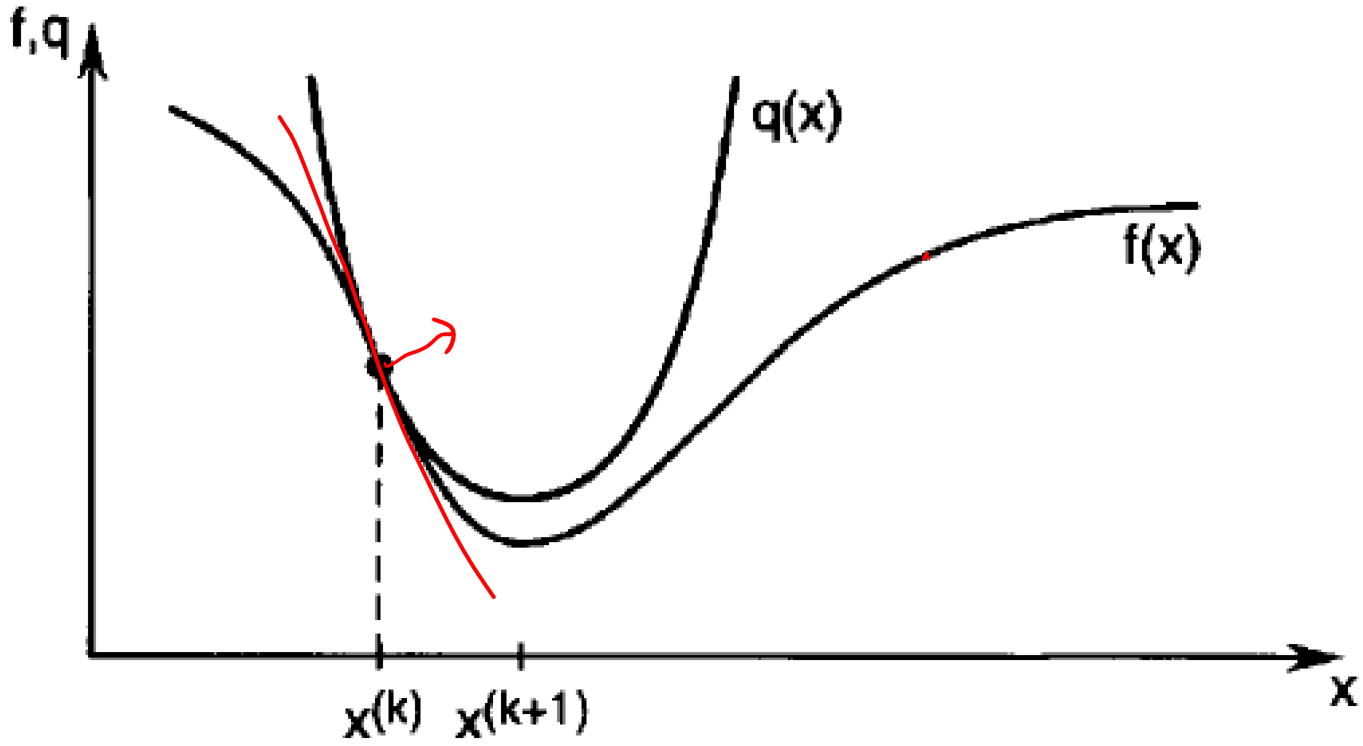
$$\therefore x^{(k+1)} = x = x^{(k)} - \frac{f'(x^{(k)})}{f''(x^{(k)})}$$

- Example: Find the minimizer of

$$f(x) = \frac{1}{2}x^2 - \cos x$$

Newton’s Method (Cont.)

- Note: Newton’s method works well when f’’(x)>0 everywhere (see Figure 7.6).



Newton's Method (Cont.)

- However, it f''(x)<0 for some x, Newton's method may fail to converge to the minimizer (see Figure 7.7).

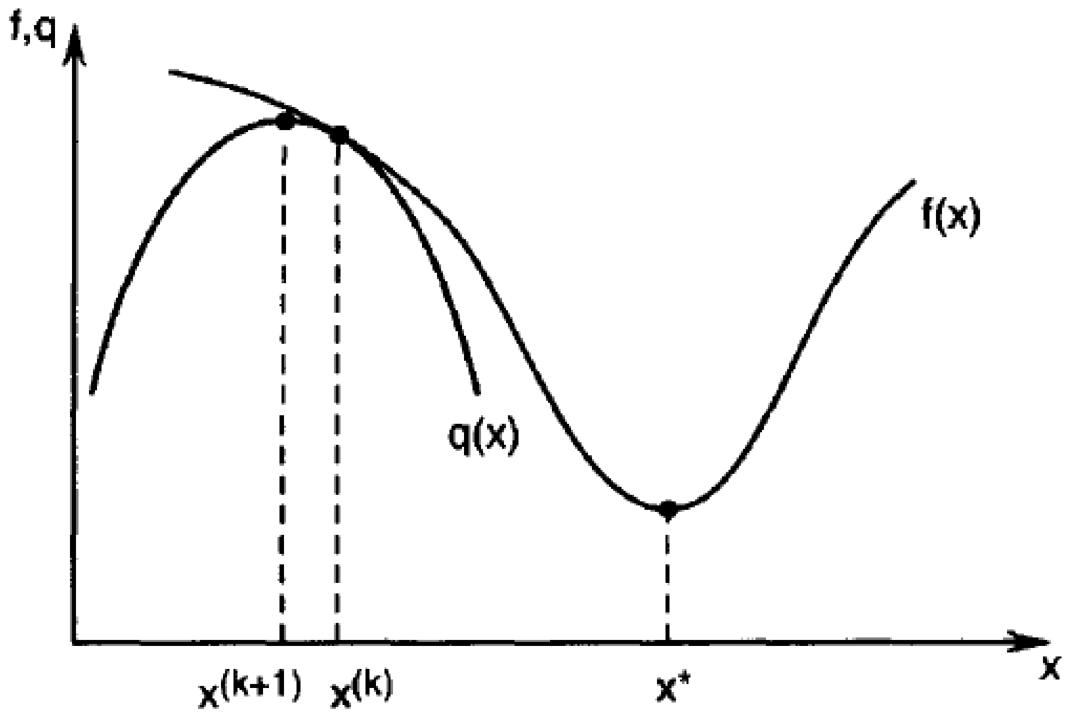


Figure 7.7 Newton's algorithm with $f''(x) < 0$

Newton's Method of Tangents

- Solving equations with Newton's method.
  - Set *g(x)=f'(x)*, we can solve equation *g(x)*=0 iteratively:

$$x^{(k+1)} = x^{(k)} - \frac{g(x^{(k)})}{g'(x^{(k)})}$$

- See Figure 7.8. Slope of g(x) at $x_{(k)}$ is

$$g'(x^{(k)}) = \frac{g(x^{(k)})}{x^{(k)} - x^{(k+1)}}.$$

$$\text{Hence, } x^{(k+1)} = x^{(k)} - \frac{g(x^{(k)})}{g'(x^{(k)})}$$

$g'(x)=0$

$$x^{(k+1)} = x^{(k)} - \frac{f'(x^{(k)})}{f''(x^{(k)})}$$

extreme

Newton's Method of Tangents (cont.)

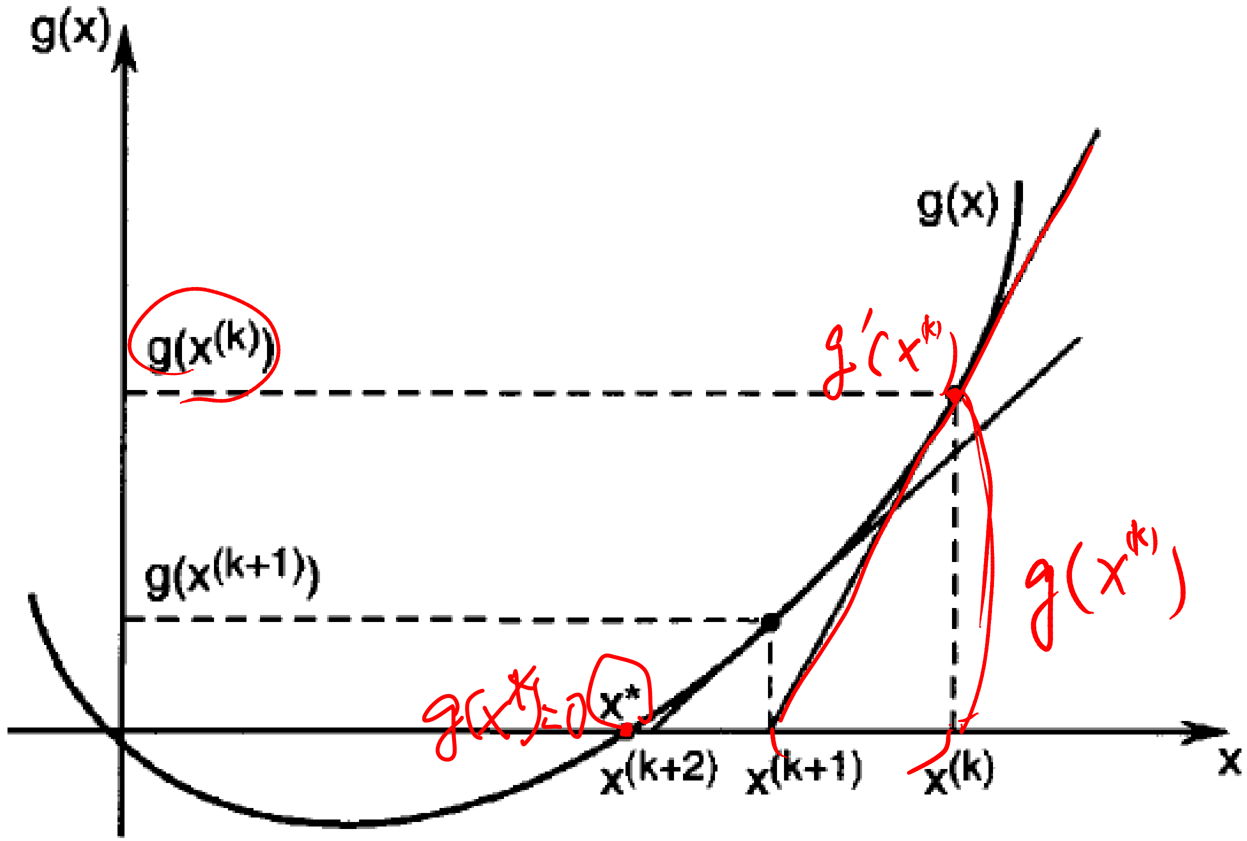


Figure 7.8 Newton's method of tangents

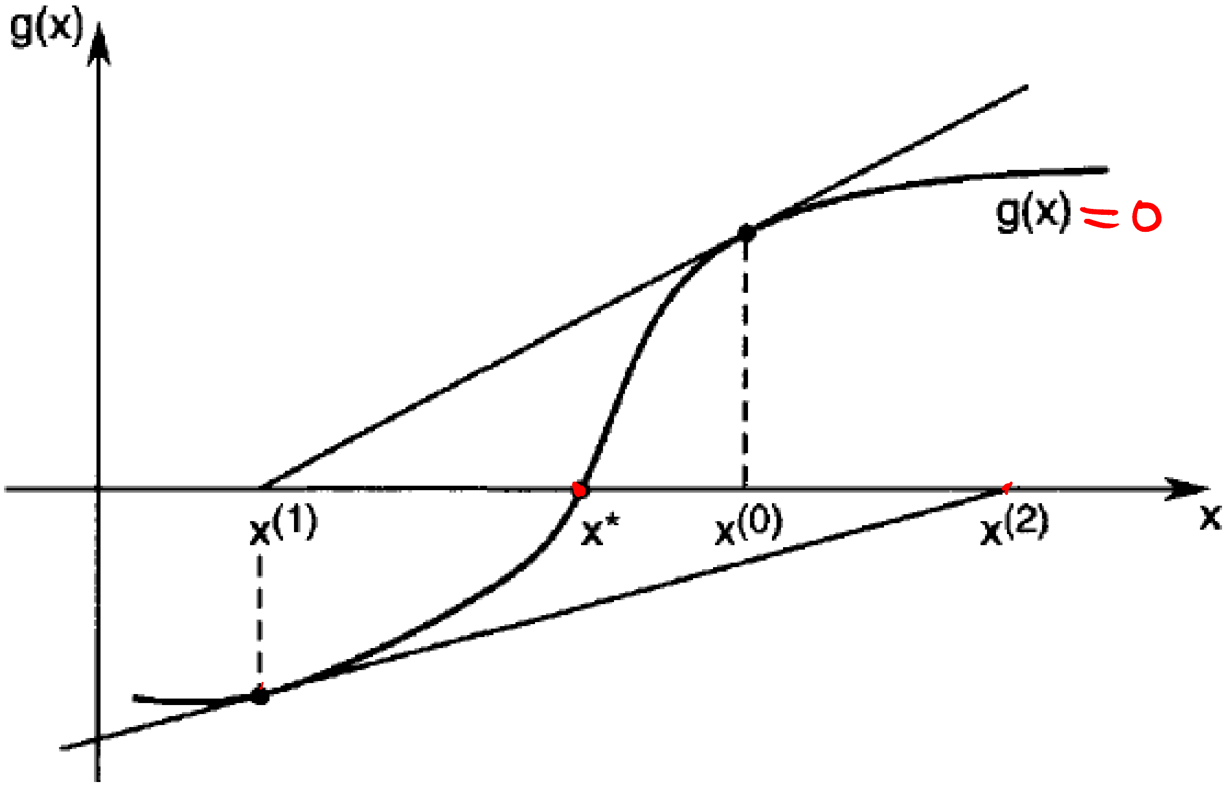
Example:

Example 7.4 We apply Newton's method to improve a first approximation, $x^{(0)} = 12$, to the root of the equation

$$g(x) = x^3 - 12.2x^2 + 7.45x + 42 = 0.$$

Newton's Method of Tangents (cont.)

- Note: Newton's method of tangents may fail if the first approximation to the root is such that the ratio $g(x_{(0)})/g'(x_{(0)})$ is not small enough (see Figure 7.9).



SECANT Method

- Approximate 2nd-order derivative of *f(x)* by

$$f''(x^{(k)}) = \frac{f'(x^{(k)}) - f'(x^{(k-1)})}{x^{(k)} - x^{(k-1)}}.$$

Thus,
Equivalently,

$$x^{(k+1)} = x^{(k)} - \frac{x^{(k)} - x^{(k-1)}}{f'(x^{(k)}) - f'(x^{(k-1)})} f'(x^{(k)}).$$

See Figure 7.10

$$x^{(k+1)} = \frac{x^{(k-1)} f'(x^{(k)}) - x^{(k)} f'(x^{(k-1)})}{f'(x^{(k)}) - f'(x^{(k-1)})}.$$

SECANT Method(cont.)

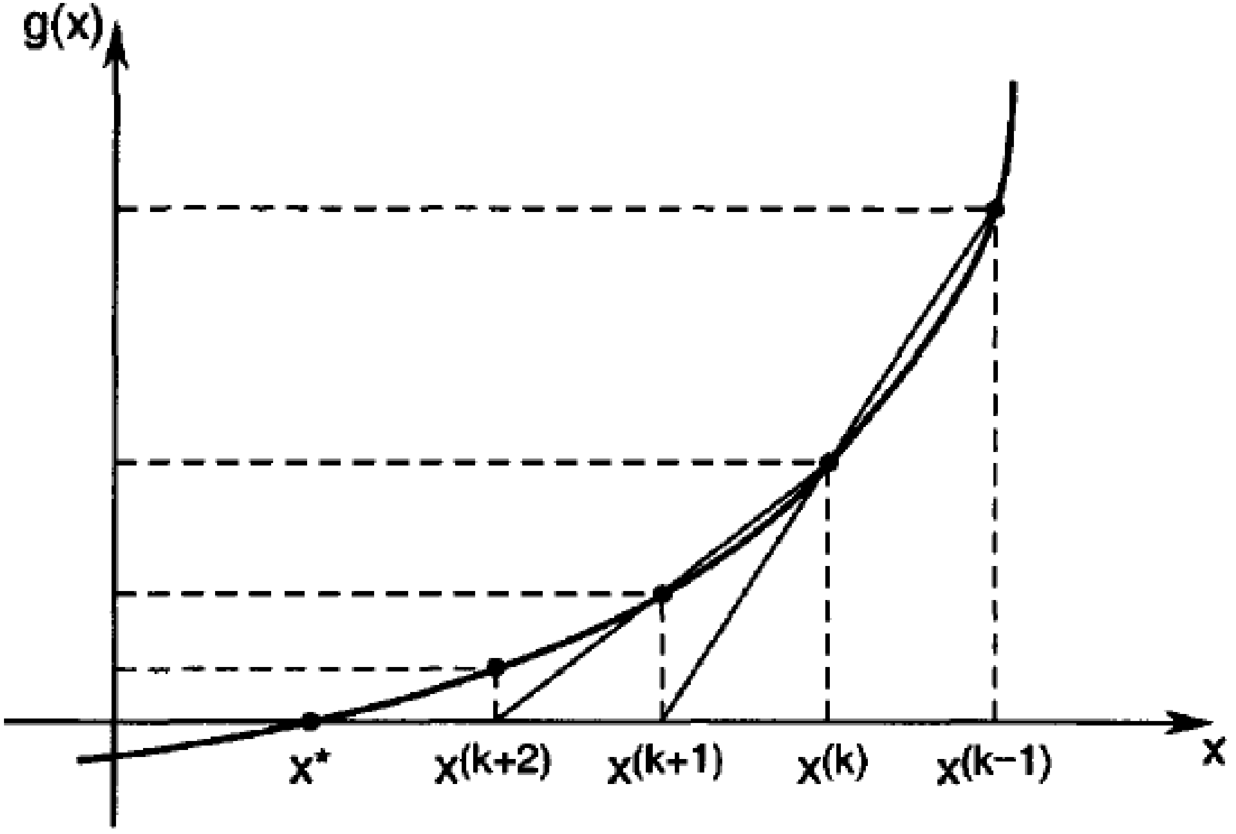


Figure 7.10 Secant method for root finding

Example:

Example 7.5 We apply the secant method to find the root of the equation

$$g(x) = x^3 - 12.2x^2 + 7.45x + 42 = 0.$$

Example 7.6 Suppose the voltage across a resistor in a circuit decays according to the model $V(t) = e^{-Rt}$, where $V(t)$ is the voltage at time t, and R is the resistance value.

Remarks

- Iterative algorithms for multidimensional optimization problem typically involve a ***line search*** at every iteration.

$$\mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} + \alpha_k \mathbf{d}^{(k)}$$

- Search direction: $\boldsymbol{d}^{(k)}$
- $\alpha^{(k)} \geq 0$ is chosen to minimize $\phi_k(\alpha) = f(\mathbf{x}^{(k)} + \alpha \mathbf{d}^{(k)})$
- Using secant method to find minimal $\phi_k(\alpha)$ needs the derivative of ϕ_k which is

$$\phi_k'(\alpha) = \mathbf{d}^{(k)^T} \nabla f(\mathbf{x}^{(k)} + \alpha \mathbf{d}^{(k)}).$$

Figure 7.11