2.4 MATRIX PRODUCTS

The *composite* of two functions: $y = sin(x)$ and $z = cos(y)$ is $z = cos(sin(x))$.

Consider two transformation systems:

$$\vec{y} = A\vec{x}, \text{ with } A = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$$

$$\vec{z} = B\vec{y}, \text{ with } B = \begin{bmatrix} 6 & 7 \\ 8 & 9 \end{bmatrix}$$

The composite of the two transformation systems is

$$\vec{z} = B(A\vec{x})$$

Question: Is $\vec{z} = T(\vec{x})$ linear? If so, what's the matrix?

(a) Find the matrix for the composite:

$$\begin{array}{l} z_1 = 6y_1 + 7y_2 \\ z_2 = 8y_1 + 9y_2 \end{array} \quad and \quad \begin{array}{rr} y_1 = & x_1 + 2x_2 \\ y_2 = & 3x_1 + 5x_2 \end{array}$$

$z_1 = 6(x_1 + 2x_2) + 7(3x_1 + 5x_2)$
$= (6 \cdot 1 + 7 \cdot 3)x_1 + (6 \cdot 2 + 7 \cdot 5)x_2$
$= 27x_1 + 47x_2$

$z_2 = 8(x_1 + 2x_2) + 9(3x_1 + 5x_2)$
$= (8 \cdot 1 + 9 \cdot 3)x_1 + (8 \cdot 2 + 9 \cdot 5)x_2$
$= 35x_1 + 61x_2$

This shows the composite is linear with matrix

$$\left[\begin{array}{cc} 6 \cdot 1 + 7 \cdot 3 & 6 \cdot 2 + 7 \cdot 5 \\ 8 \cdot 1 + 9 \cdot 3 & 8 \cdot 2 + 9 \cdot 5 \end{array} \right] = \left[\begin{array}{cc} 27 & 47 \\ 35 & 61 \end{array} \right]$$

(b)Use Fact to show the transformation $T(\vec{x}) = B(A\vec{x})$ is linear:

$T(\vec{v}+\vec{w}) = B(A(\vec{v}+\vec{w})) = B(A\vec{v}+A\vec{w}) = B(A\vec{v}) + B(A\vec{w}) = T(\vec{v}) + T(\vec{w})$

$T(k\vec{v}) = B(A(k\vec{v})) = B(k(A\vec{v})) = k(B(A\vec{v})) = kT(\vec{v})$

Once we know that T is linear, we can find its matrix by computing the vectors: $T(\vec{e}_1)$ and $T(\vec{e}_2)$:

$T(\vec{e}_1) = B(A(\vec{e}_1))$=$B$(first column of A) = $\begin{bmatrix} 6 & 7 \\ 8 & 9 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 27 \\ 35 \end{bmatrix}$

$T(\vec{e}_2) = B(A(\vec{e}_1))$=$B$(second column of A) = $\begin{bmatrix} 6 & 7 \\ 8 & 9 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 47 \\ 61 \end{bmatrix}$

The matrix of $T(\vec{x}) = B(A\vec{x}) = BA(\vec{x})$:

$$= \begin{bmatrix} | & | \\ T(\vec{e}_1) & T(\vec{e}_2) \\ | & | \end{bmatrix} = \begin{bmatrix} 27 & 47 \\ 35 & 61 \end{bmatrix}$$

Definition. Matrix multiplication

1. Let B be an $m \times n$ matrix and A a $q \times p$ matrix. The product BA is defined if (and only if) $n = q$.

2. If B is an $m \times n$ matrix and A an $n \times p$ matrix, then the product BA is defined as the matrix of the linear transformation $T(\vec{x}) = B(A\vec{x})$. This means that $T(\vec{x}) = B(A\vec{x}) = (BA)\vec{x}$, for all $\vec{x}$ in R^p. The product BA is an $m \times p$ matrix.

Let B be an $m \times n$ matrix and A an $n \times p$ matrix. Let's think about the columns of the matrix BA:

(ith columns of BA)
$= (BA)\vec{e}_i$
$= B(A\vec{e}_i)$
$= B$(ith column of A).

If we denote the columns of A by $\vec{v}_1, \vec{v}_2, ..., \vec{v}_p$,we can write

$$BA = B \underbrace{\begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \ldots & \vec{v}_p \\ | & | & & | \end{bmatrix}}_{A} =$$

$$\begin{bmatrix} | & | & & | \\ B\vec{v}_1 & B\vec{v}_2 & \ldots & B\vec{v}_p \\ | & | & & | \end{bmatrix}.$$

The matrix product, column by column

Let B be an $m \times n$ matrix and A an $n \times p$ matrix with columns $\vec{v}_1, \vec{v}_2, ..., \vec{v}_p$. Then, the product BA is

$$BA = B \begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \dots & \vec{v}_p \\ | & | & & | \end{bmatrix} = \begin{bmatrix} | & | & & | \\ B\vec{v}_1 & B\vec{v}_2 & \dots & B\vec{v}_p \\ | & | & & | \end{bmatrix}.$$

To find BA, we can multiply B with the columns of A and combine the resulting vectors.

Fact Matrix multiplication is noncommutative: $AB \neq BA$, in general. However, at times it does happen that $AB = BA$; then, we say that the matrices A and B commute.

The matrix product, entry by entry

Let B be an $m \times n$ matrix and A an $n \times p$ matrix. The ijth entry of BA is the dot product of the ith row of B and the jth column of A.

$$\begin{bmatrix} b_{11} & b_{12} & \ldots & b_{1n} \\ b_{21} & b_{22} & \ldots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{i1} & b_{i2} & \ldots & b_{in} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \ldots & b_{mn} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1j} & \ldots & a_{1p} \\ a_{21} & a_{22} & \ldots & a_{2j} & \ldots & a_{2p} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nj} & \ldots & a_{np} \end{bmatrix}$$

is the $m \times p$ matrix whose ijth entry is

$$b_{i1}a_{1j} + b_{i2}a_{2j} + \ldots + b_{in}a_{nj} = \sum_{k=1}^{n} b_{ik}a_{kj}.$$

Example. $\begin{bmatrix} 6 & 7 \\ 8 & 9 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} =$

$$\begin{bmatrix} 6 \cdot 1 + 7 \cdot 3 & 6 \cdot 2 + 7 \cdot 5 \\ 8 \cdot 1 + 9 \cdot 3 & 8 \cdot 2 + 9 \cdot 5 \end{bmatrix} = \begin{bmatrix} 27 & 47 \\ 35 & 61 \end{bmatrix}.$$

We have done these computations before.(where?)

Matrix Algebra

Fact (a) For an invertible $n \times n$ matrix A.

$$AA^{-1} = I_n \text{ and } A^{-1}A = I_n.$$

Fact (b) For an $m \times n$ matrix A.

$$AI_n = I_mA = A.$$

Fact (c) Matrix multiplication is associative

$$(AB)C = A(BC).$$

We can write simply ABC for the product $(AB)C = A(BC)$.

Proof (a) $(AB)C = (AB)[\ \vec{v}_1 \ \ \vec{v}_2 \ \ \cdots \ \ \vec{v}_q\ ]$

$$= [\ (AB)\vec{v}_1 \ \ (AB)\vec{v}_2 \ \ \cdots \ \ (AB)\vec{v}_q\ ]$$

and

$$A(BC) = A[\ B\vec{v}_1 \ \ B\vec{v}_2 \ \ \cdots \ \ B\vec{v}_q\ ]$$

$$= [\ A(B\vec{v}_1) \ \ A(B\vec{v}_2) \ \ \cdots \ \ A(B\vec{v}_q)\ ]$$

Since $(AB)\vec{v}_i = A(B\vec{v}_i)$, by definition of the matrix product, we find that $(AB)C = A(BC)$.

Proof (b) Consider two linear transformations

$$T(\vec{x}) = ((AB)C)\vec{x}$$

and

$$L(\vec{x}) = (A(BC))\vec{x}$$

are identical because,

$$T(\vec{x}) = ((AB)C)\vec{x} = (AB)(C\vec{x}) = A(B(C\vec{x}))$$

and

$$L(\vec{x}) = (A(BC))\vec{x} = A((BC)\vec{x}) = A(B(C\vec{x}))$$

If A and B are invertible $n \times n$ matrices, is BA invertible?

$$\vec{y} = BA\vec{x}$$

multiply both sides by B^{-1}

$$B^{-1}\vec{y} = B^{-1}BA\vec{x} = I_n A\vec{x} = A\vec{x}$$

next, multiply both sides by A^{-1}

$$A^{-1}B^{-1}\vec{y} = A^{-1}A\vec{x} = I_n\vec{x} = \vec{x}$$

This computation shows that the linear transformation is invertible since

$$\vec{x} = A^{-1}B^{-1}\vec{y}$$

Fact The inverse of a product of matrices

If A and B are invertible $n \times n$ matrices, then BA is invertible as well, and

$$(BA)^{-1} = A^{-1}B^{-1}.$$

Pay attention to the order of the matrices.

Proof Verify it by yourself.

Fact Let A and B be two $n \times n$ matrices such that

$$BA = I_n.$$

Then,

a. A and B are both invertible.
b. $A^{-1} = B$ and $B^{-1} = A$, and
c. $AB = I_n$.

Proof (a) To demonstrate A is invertible it suffices to show that the linear system $A\vec{x} = \vec{0}$ has only the solution $\vec{x} = \vec{0}$.

$$BA\vec{x} = B\vec{0} = \vec{0}$$

(b) $B = A^{-1}$ since

$$(BA)A^{-1} = (I_n)A^{-1} = A^{-1}$$

and

$$B^{-1} = (A^{-1})^{-1} = A$$

(c) $AB = AA^{-1} = I_n$

Example. *B*= $\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ *is the inverse of* $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

it suffices to verify that $BA = I_2$*:*

$$BA = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$= \frac{1}{ad-bc}\begin{bmatrix} ad-bc & bd-bd \\ ac-ac & ad-bc \end{bmatrix} = I_2.$$

Example.

Suppose A, B and C are three $n \times n$ matrices and $ABC = I_n$. Show that B is invertible, and express B^{-1} in term of A and C.

Solution

Write $ABC = (AB)C = I_n$. We have $C(AB) = I_n$. Since matrix multiplication is associative, we can write $(CA)B = I_n$. We conclude that B is invertible, and $B^{-1} = CA$.

Distributive property for matrices

Fact If A, B are $n \times n$, and C, D are $n \times p$ matrices, then

$$A(C + D) = AC + AD$$

and

$$(A + B)C = AC + BC.$$

Fact If A is an $m \times n$ matrix, B an $n \times p$ matrix, and k a scalar, then

$$(kA)B = A(kB) = k(AB).$$

Partitioned Matrices

It is sometimes useful to break a large matrix down into smaller submatrices by slicing it up with horizontal or vertical lines that go all the way through the matrix.

For example, we can think of the 4×4 matrix

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 8 & 7 & 6 \\ 5 & 4 & 3 & 2 \end{bmatrix}$$

as a 2×2 matrix whose "entries" are four 2×2 matrices:

$$A = \left[\begin{array}{cc|cc} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ \hline 9 & 8 & 7 & 6 \\ 5 & 4 & 3 & 2 \end{array}\right] = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

with $A_{11} = \begin{bmatrix} 1 & 2 \\ 5 & 7 \end{bmatrix}$, $A_{12} = \begin{bmatrix} 3 & 4 \\ 7 & 8 \end{bmatrix}$, etc.

The submatrices in such a partition need not be of equal size; for example, we could have

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 4 & 5 & 6 \\ \hline 7 & 8 & 9 \end{array}\right] = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}.$$

A useful property of partitioned matrices is the following:

Multiplying partitioned matrices Partitioned matrices can be multiplied as though the submatrices were scalars:

$AB =$

$$\begin{bmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ A_{21} & A_{22} & \dots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{i1} & A_{i2} & \dots & A_{in} \\ \vdots & \vdots & \ddots & \vdots \\ A_{m1} & A_{m2} & \dots & A_{mn} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} & \dots & B_{1j} & \dots & B_{1p} \\ B_{21} & B_{22} & \dots & B_{2j} & \dots & B_{2p} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ B_{n1} & B_{n2} & \dots & B_{nj} & \dots & B_{np} \end{bmatrix}$$

is the partitioned matrix whose ijth "entry" is the matrix

$$A_{i1}B_{1j} + A_{i2}B_{2j} + \ldots + A_{in}B_{nj} = \sum_{k=1}^{n} A_{ik}B_{kj},$$

provided that all the products $A_{ik}B_{kj}$ are defined.

Example.

$$\left[\begin{array}{cc|c} 0 & 1 & -1 \\ 1 & 0 & 1 \end{array}\right] \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 4 & 5 & 6 \\ \hline 7 & 8 & 9 \end{array}\right]$$

$$= \left[\begin{array}{c|c} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \end{bmatrix}[\,7 \quad 8\,] & \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 3 \\ 6 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \end{bmatrix}[9] \end{array}\right]$$

$$= \left[\begin{array}{cc|c} -3 & -3 & -3 \\ 8 & 10 & 12 \end{array}\right].$$

Compute this product without using a partition, and see whether you find the same result.

Example.

$$A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix},$$

where A_{11} is an $n \times n$ matrix, A_{22} is an $m \times m$ matrix, and A_{12} is an $n \times m$ matrix.

a. For which choices of A_{11}, A_{12}, and A_{22} is A invertible ?

b. If A is invertible, what is A^{-1} (in terms of A_{11}, A_{12}, A_{22})?

Solution

We are looking for an $(n+m)\times(n+m)$ matrix B such that

$$BA = I_{n+m} = \begin{bmatrix} I_n & 0 \\ 0 & I_m \end{bmatrix}.$$

Let us partition B in the same way as A:

$$B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix},$$

where B_{11} is $n\times n$, B_{22} is $m\times m$, etc. The fact that B is the inverse of A means that

$$\begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ 0 & B_{22} \end{bmatrix} = \begin{bmatrix} I_n & 0 \\ 0 & I_m \end{bmatrix},$$

or using

$$\left| \begin{array}{rrcr} & B_{11}A_{11} & = & I_n \\ B_{11}A_{12} & +B_{12}A_{22} & = & 0 \\ & B_{21}A_{11} & = & 0 \\ B_{21}A_{12} & +B_{22}A_{22} & = & I_m \end{array} \right| .$$

We have to solve this system for the submatrices B_{ij}.

1. By Equation 1, A_{11} must be invertible, and $B_{11} = A_{11}^{-1}$.

2. By Equation 3, $B_{21} = 0$ (Multiply by A_{11}^{-1} form the right)

3. Equation 4 now simplifies to $B_{22}A_{22} = I_m$. Therefore, A_{22} must be invertible, and $B_{22} = A_{22}^{-1}$.

4. Lastly, Solve for B_{12} by Equation 2

$$A_{11}^{-1}A_{12} + B_{12}A_{22} = 0$$

$$\Rightarrow B_{12}A_{22} = -A_{11}^{-1}A_{12}$$

$$\Rightarrow B_{12} = -A_{11}^{-1}A_{12}A_{22}^{-1}$$

So

a. A is invertible if (and only if) both A_{11} and A_{22} are invertible (no condition is imposed on A_{12}).

b. If A is invertible, then its inverse is

$$A^{-1} = \begin{bmatrix} A_{11}^{-1} & -A_{11}^{-1}A_{12}A_{22}^{-1} \\ 0 & A_{22}^{-1} \end{bmatrix}.$$

Verify this result for the following example:

Example. *5*

$$\left[\begin{array}{cc|ccc} 1 & 1 & 1 & 2 & 3 \\ 1 & 2 & 4 & 5 & 6 \\ \hline 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]^{-1} = \left[\begin{array}{cc|ccc} 2 & -1 & 2 & 1 & 0 \\ -1 & 1 & -3 & -3 & -3 \\ \hline 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right].$$

Homework.

Exercise 2.4: 5, 13, 17, 23, 27, 35