Applied Linear Algebra

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Chapter 2
Linear Transformation

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2.1 Introduction to Linear Transformations and Their Inverse

See Figure 1

Encryption of a coordinate $\vec{x} = \begin{bmatrix} 5 \\ 42 \end{bmatrix}$ to $\vec{y}$ by the following code:

$$\begin{aligned} y_1 &= \quad x_1 + \ 3x_2 = 131 \\ y_2 &= \ 2x_1 + \ 5x_2 = 220 \end{aligned}$$

At the headquarter, $\vec{y} = \begin{bmatrix} 131 \\ 220 \end{bmatrix}$ is received. We need to determine the actual $\vec{x}$ by solve the linear system.

$$A\vec{x} = \vec{b}$$

i.e.
$$\begin{aligned} x_1 + \ 3x_2 &= 131 \\ 2x_1 + \ 5x_2 &= 220 \end{aligned}$$

If $\vec{y} = \begin{bmatrix} 133 \\ 223 \end{bmatrix}$ We need to solve it again by:

$$\begin{aligned} x_1 + \ 3x_2 &= 133 \\ 2x_1 + \ 5x_2 &= 223 \end{aligned}$$

For a general formula, we need solve the system

$$\begin{aligned} x_1 + \ 3x_2 &= y_1 \\ 2x_1 + \ 5x_2 &= y_2 \end{aligned}$$

for arbitrary constants y_1 and y_2.

For sender: $\vec{x} \to \vec{y}$ (encoding)

For receiver: $\vec{y} \to \vec{x}$ (decoding)

$$\left|\begin{array}{rl} x_1 + 3x_2 = y_1 & \\ 2x_1 + 5x_2 = & y_2 \end{array}\right| \begin{array}{r} \longrightarrow \\ -2(I) \end{array}$$

$$\left|\begin{array}{rl} x_1 + 3x_2 & = y_1 \\ -x_2 & = -2y_1 + y_2 \end{array}\right| \begin{array}{r} \longrightarrow \\ \div(-1) \end{array}$$

$$\left|\begin{array}{rl} x_1 + 3x_2 & = y_1 \\ x_2 & = 2y_1 - y_2 \end{array}\right| \begin{array}{r} -3(II) \\ \longrightarrow \end{array}$$

$$\left|\begin{array}{lcrr} x_1 & = & -5y_1 & +3y_2 \\ \quad x_2 & = & 2y_1 & -y_2 \end{array}\right|$$

The decoding formula is:

$$\begin{array}{lrr} x_1 = & -5y_1 & +3y_2 \\ x_2 = & 2y_1 & -y_2 \end{array}$$

or $\vec{x} = B\vec{y}$, where $B = \begin{bmatrix} -5 & 3 \\ 2 & -1 \end{bmatrix}$

Definition. *We say that the matrix B is the* **inverse** *of the matrix A and write $B = A^{-1}$.*

$$\vec{x} \underset{\vec{x}=B\vec{y},\, B=\begin{bmatrix} -5 & 3 \\ 2 & -1 \end{bmatrix}}{\overset{\vec{y}=A\vec{x},\, A=\begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}}{\rightleftarrows}} \vec{y}$$

The coding transformation is represented as

$$\underbrace{\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}}_{\vec{y}} = \begin{bmatrix} x_1 + & 3x_2 \\ 2x_1 + & 5x_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\vec{x}}$$

or succinctly, as $\boxed{\vec{y} = A\vec{x}}$.

A transformation of the form $\vec{y} = A\vec{x}$ is called a **linear transformation**.

Function: Consider two sets X and Y. A function $T : X \to Y$ is a rule that associates with each element $x \in X$ a unique element $y \in Y$.

The set X is called the *domain* and Y is called its *codomain*.

Definition. *A function T from R^n to R^m is called a* **linear transformation** *if there is an $m \times n$ matrix A such that*

$T(\vec{x}) = A\vec{x}$, *for all $\vec{x}$ in R^n.*

Example. *The linear transformation system*

$$\begin{aligned} y_1 &= 7x_1 + 3x_2 - 9x_3 + 8x_4 \\ y_2 &= 6x_1 + 2x_2 - 8x_3 + 7x_4 \\ y_3 &= 8x_1 + 4x_2 \qquad\quad + 7x_4 \end{aligned}$$

(a function from R^4 to R^3) can be represented by the 3×4 matrix

$$A = \begin{bmatrix} 7 & 3 & -9 & 8 \\ 6 & 2 & -8 & 7 \\ 8 & 4 & 0 & 7 \end{bmatrix}$$

Example. *The identity transformation system*

$$\begin{aligned} y_1 &= x_1 \\ y_2 &= x_2 \\ &\vdots \\ y_n &= x_n \end{aligned}$$

(a linear transformation from R^n *to* R^n *whose output equals its input) is represented by* $n \times n$ *matrix*

$$A = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

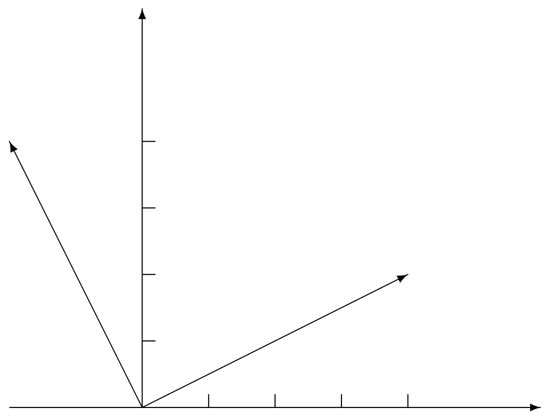
This matrix is called the **identiy matrix** *and is denoted by* I_n*:*

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \; I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \; etc.$$

Example. *Give a geometric interpretation of the linear transformation*

$$\vec{y} = A\vec{x}, \text{ where } A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix}.$$



See Figure 4 (pp.45).

Fact 2.1.2 Consider a linear transformation T from R^n to R^m. Let $\vec{e}_i = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \leftarrow ith$

The matrix of T can be represented as

$$A = \begin{bmatrix} | & | & & | \\ T(\vec{e}_1) & T(\vec{e}_2) & \cdots & T(\vec{e}_n) \\ | & | & & | \end{bmatrix}$$

Since

$$T(\vec{e}_i) = A\vec{e}_i = \begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} = \vec{v}_i$$

Example. *Find the inverse for the following matrix:*

$$\begin{bmatrix} 1 & 2 \\ 3 & 9 \end{bmatrix}$$

Solution

$$\left[\begin{array}{cccc} 1 & 2 & \vdots & y_1 \\ 3 & 9 & \vdots & y_2 \end{array}\right] -3(I)$$

$$\longrightarrow \left[\begin{array}{cccc} 1 & 2 & \vdots & y_1 \\ 0 & 3 & \vdots & -3y_1 + y_2 \end{array}\right] \div 3$$

$$\longrightarrow \left[\begin{array}{cccc} 1 & 2 & \vdots & y_1 \\ 0 & 1 & \vdots & -y_1 + \frac{1}{3}y_2 \end{array}\right] -2(II)$$

$$\longrightarrow \left[\begin{array}{cccc} 1 & 0 & \vdots & 3y_1 - \frac{2}{3}y_2 \\ 0 & 1 & \vdots & -y_1 + \frac{1}{3}y_2 \end{array}\right]$$

$$\Rightarrow \begin{bmatrix} 1 & 2 \\ 3 & 9 \end{bmatrix}^{-1} = \begin{bmatrix} 3 & -\frac{2}{3} \\ -1 & \frac{1}{3} \end{bmatrix}$$

Example. *Find the inverse for the following matrix:* $\begin{bmatrix} 3 & -\frac{2}{3} \\ -1 & \frac{1}{3} \end{bmatrix}$

Solution

$$\left[\begin{array}{cc:c} 3 & -\frac{2}{3} & y_1 \\ -1 & \frac{1}{3} & y_2 \end{array}\right] \div 3$$

$$\longrightarrow \left[\begin{array}{cc:c} 1 & -\frac{2}{9} & \frac{1}{3}y_1 \\ -1 & \frac{1}{3} & y_2 \end{array}\right] +(I)$$

$$\longrightarrow \left[\begin{array}{cc:c} 1 & -\frac{2}{9} & \frac{1}{3}y_1 \\ 0 & \frac{1}{9} & \frac{1}{3}y_1 + y_2 \end{array}\right] \times 9$$

$$\longrightarrow \left[\begin{array}{cc:c} 1 & -\frac{2}{9} & \frac{1}{3}y_1 \\ 0 & 1 & 3y_1 + 9y_2 \end{array}\right] +\frac{2}{9}(II)$$

$$\longrightarrow \left[\begin{array}{cc:c} 1 & 0 & y_1 + 2y_2 \\ 0 & 1 & 3y_1 + 9y_2 \end{array}\right]$$

$$\Rightarrow \begin{bmatrix} 3 & -\frac{2}{3} \\ -1 & \frac{1}{3} \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 2 \\ 3 & 9 \end{bmatrix}$$

Example. *Not all linear transformations are invertible. Consider the matrix* $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

If $\vec{y} = \begin{bmatrix} 89 \\ 178 \end{bmatrix}$, to solve the system

$$\left| \begin{array}{rrr} x_1 & +2x_2 = & 89 \\ 2x_1 & +4x_2 = & 178 \end{array} \right|$$

We discover there are infinitely many solutions

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 89 - 2t \\ t \end{bmatrix}$$

We say that the coding matrix A are *noninvertible*.

Homework. *Exercises 2.1: 4, 5, 7, 10, 12, 15*